Representable Commutative Atoms in the Subvariety Lattice of Residuated Lattices

Rostislav Horčík

Institute of Computer Science
Academy of Sciences of the Czech Republic

Shanks Workshop on Ordered Groups in Logic
Vanderbilt, Nashville, 2009

Introduction

- The notions of a nucleus and a conucleus turned out to be quite useful in the theory of residuated lattices.

Introduction

- The notions of a nucleus and a conucleus turned out to be quite useful in the theory of residuated lattices.
- Prominent examples showing this are e.g. the results on categorical equivalences between several categories of residuated lattices and ℓ-groups endowed with an operator which is a composition of a conucleus and a nucleus.

Introduction

- The notions of a nucleus and a conucleus turned out to be quite useful in the theory of residuated lattices.
- Prominent examples showing this are e.g. the results on categorical equivalences between several categories of residuated lattices and ℓ-groups endowed with an operator which is a composition of a conucleus and a nucleus.
- The above-mentioned examples shows that various residuated lattices can be constructed from ℓ-groups by means of conuclei and nuclei. Moreover, it seems that the class of residuated lattices which can be obtained in this way is quite large.

Introduction

- The notions of a nucleus and a conucleus turned out to be quite useful in the theory of residuated lattices.
- Prominent examples showing this are e.g. the results on categorical equivalences between several categories of residuated lattices and ℓ-groups endowed with an operator which is a composition of a conucleus and a nucleus.
- The above-mentioned examples shows that various residuated lattices can be constructed from ℓ-groups by means of conuclei and nuclei. Moreover, it seems that the class of residuated lattices which can be obtained in this way is quite large.
- In this talk we construct some algebras belonging to this class.

Motivation

- In the recent book by Galatos, Jipsen, Kowalski, and Ono it is conjectured that there are only countably many representable commutative atoms in $\Lambda(\mathsf{RL})$.

Motivation

- In the recent book by Galatos, Jipsen, Kowalski, and Ono it is conjectured that there are only countably many representable commutative atoms in $\Lambda(\mathsf{RL})$.
- In order to prove this conjecture, it was sufficient to understand the 1-generated integral commutative residuated chains and show that there are only countably many of them.

Motivation

- In the recent book by Galatos, Jipsen, Kowalski, and Ono it is conjectured that there are only countably many representable commutative atoms in $\Lambda(\mathsf{RL})$.
- In order to prove this conjecture, it was sufficient to understand the 1-generated integral commutative residuated chains and show that there are only countably many of them.
- Unfortunately, this turned out to be a false expectation.

Motivation

- In the recent book by Galatos, Jipsen, Kowalski, and Ono it is conjectured that there are only countably many representable commutative atoms in $\Lambda(\mathsf{RL})$.
- In order to prove this conjecture, it was sufficient to understand the 1-generated integral commutative residuated chains and show that there are only countably many of them.
- Unfortunately, this turned out to be a false expectation.
- We construct uncountable many 1-generated integral commutative residuated chains which can be easily modified so that they generate representable commutative atoms in $\Lambda(\mathsf{RL})$.

Results on $\Lambda(\mathsf{RL})$

We will start with a totally ordered Abelian ℓ-group and using conuclear contractions and nuclear retractions we will prove the following results:

Results on $\Lambda(\mathsf{RL})$

We will start with a totally ordered Abelian ℓ-group and using conuclear contractions and nuclear retractions we will prove the following results:

- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{RL})$.

Results on $\Lambda(\mathsf{RL})$

We will start with a totally ordered Abelian ℓ-group and using conuclear contractions and nuclear retractions we will prove the following results:

- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{RL})$.
- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{FL_{ei}})$.

Results on $\Lambda(\mathsf{RL})$

We will start with a totally ordered Abelian ℓ-group and using conuclear contractions and nuclear retractions we will prove the following results:

- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{RL})$.
- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{FL_{ei}})$.
- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{FL_{eo}})$.

Results on $\Lambda(\mathsf{RL})$

We will start with a totally ordered Abelian ℓ-group and using conuclear contractions and nuclear retractions we will prove the following results:

- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{RL})$.
- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{FL_{ei}})$.
- There are $2^{\aleph_0}$ representable commutative atoms in $\Lambda(\mathsf{FL_{eo}})$.

On the other hand, we also prove the following result:

- There are 2^1 representable commutative integral atoms in $\Lambda(\mathsf{RL})$.

Nucleus and conucleus

Definition

- A closure operator γ on a commutative residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \rightarrow, 1\rangle$ is called a nucleus if $\gamma(x)\gamma(y) \leq \gamma(xy)$.

Nucleus and conucleus

Definition

- A closure operator γ on a commutative residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \rightarrow, 1 \rangle$ is called a nucleus if $\gamma(x)\gamma(y) \leq \gamma(xy)$.
- An interior operator σ on a commutative residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \rightarrow, 1 \rangle$ is called a conucleus if $\sigma(1) = 1$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$.

Nucleus and conucleus

Definition

- A closure operator γ on a commutative residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \rightarrow, 1\rangle$ is called a nucleus if $\gamma(x)\gamma(y) \leq \gamma(xy)$.
- An interior operator σ on a commutative residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \rightarrow, 1\rangle$ is called a conucleus if $\sigma(1) = 1$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$.

Let $\gamma : L \rightarrow L$ be an operator on L. The image of γ is denoted L_γ.

Closure retraction and interior extraction

Lemma

- *An operator γ on $\mathbf{L}$ is nucleus iff L_γ satisfies*

$$\min\{a \in L_\gamma \mid x \leq a\} \text{ exists for all } x \in L.$$

and

$$x \to y \in L_\gamma \text{ for all } x \in L \text{ and } y \in L_\gamma.$$

L_γ is called nuclear (closure) retraction.

Closure retraction and interior extraction

Lemma

- *An operator γ on $\mathbf{L}$ is nucleus iff L_γ satisfies*

  $$\min\{a \in L_\gamma \mid x \leq a\} \text{ exists for all } x \in L.$$

  and

  $$x \to y \in L_\gamma \text{ for all } x \in L \text{ and } y \in L_\gamma.$$

  L_γ is called nuclear (closure) retraction.

- *An operator σ on $\mathbf{L}$ is conucleus iff L_σ is a submonoid of $\mathbf{L}$ and*

  $$\max\{a \in L_\sigma \mid a \leq x\} \text{ exists for all } x \in L.$$

  L_σ is called conuclear (interior) contraction.

Resulting residuated algebras

Lemma

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, 1\rangle$ *is a commutative residuated lattice and* γ *a nucleus on it, then the algebra* $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \to, \gamma(1)\rangle$ *is a commutative residuated lattice, where* $x \vee_\gamma y = \gamma(x \vee y)$ *and* $x \circ_\gamma y = \gamma(x \cdot y)$*, for all* $x, y \in L_\gamma$*.*

Resulting residuated algebras

Lemma

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, 1\rangle$ *is a commutative residuated lattice and* γ *a nucleus on it, then the algebra* $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \to, \gamma(1)\rangle$ *is a commutative residuated lattice, where* $x \vee_\gamma y = \gamma(x \vee y)$ *and* $x \circ_\gamma y = \gamma(x \cdot y)$*, for all* $x, y \in L_\gamma$*.*

Lemma

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, 1\rangle$ *is a commutative residuated lattice and* σ *a conucleus on it, then the algebra* $\mathbf{L}_\sigma = \langle L_\sigma, \wedge_\sigma, \vee, \cdot, \to_\sigma, 1\rangle$ *is a residuated lattice, where* $x \wedge_\sigma y = \sigma(x \wedge y)$ *and* $x \to_\sigma y = \sigma(x \to y)$*, for all* $x, y \in L_\sigma$*.*

Construction of $\mathbf{A}_S$

- Let $\mathbf{G}$ be the lexicographic product of two copies of $\mathbf{Z}$.

Construction of $\mathbf{A}_S$

- Let $\mathbf{G}$ be the lexicographic product of two copies of $\mathbf{Z}$.
- For each infinite $S \subseteq -2 - \mathbb{N}$ we will construct a residuated chain $\mathbf{A}_S$ by means of a conucleus σ_S and a nucleus γ.

Construction of $\mathbf{A}_S$

- Let $\mathbf{G}$ be the lexicographic product of two copies of $\mathbf{Z}$.
- For each infinite $S \subseteq -2 - \mathbb{N}$ we will construct a residuated chain $\mathbf{A}_S$ by means of a conucleus σ_S and a nucleus γ.
- The conucleus σ_S is defined by its conuclear contraction:

$$G_{\sigma_S} = \{\langle 0,0\rangle, \langle -1,0\rangle, \langle -1,-1\rangle\} \cup \{\langle -1,z\rangle \in A \mid z \in S\} \cup \{\langle x,y\rangle \in A \mid x \leq -2\}.$$

Construction of $\mathbf{A}_S$

- Let $\mathbf{G}$ be the lexicographic product of two copies of $\mathbf{Z}$.
- For each infinite $S \subseteq -2 - \mathbb{N}$ we will construct a residuated chain $\mathbf{A}_S$ by means of a conucleus σ_S and a nucleus γ.
- The conucleus σ_S is defined by its conuclear contraction:

$$G_{\sigma_S} = \{\langle 0,0\rangle, \langle -1,0\rangle, \langle -1,-1\rangle\} \cup \{\langle -1,z\rangle \in A \mid z \in S\} \cup \{\langle x,y\rangle \in A \mid x \leq -2\}.$$

- Since S is infinite and dually well ordered, we get the following lemma.

Lemma

The set G_{σ_S} forms a conuclear contraction.

Construction of $\mathbf{A}_S$ (cont.)

- Next, we define the nucleus $\gamma(x, y) = \langle x, y\rangle \vee \langle -3, -1\rangle$.

Construction of $\mathbf{A}_S$ (cont.)

- Next, we define the nucleus $\gamma(x, y) = \langle x, y\rangle \vee \langle -3, -1\rangle$.
- Then $\mathbf{A}_S$ is the subalgebra of $(\mathbf{G}_{\sigma_S})_\gamma$ generated by $\{a = \langle -1, 0\rangle, b = \langle -1, -1\rangle\}$.

Construction of $\mathbf{A}_S$ (cont.)

- Next, we define the nucleus $\gamma(x, y) = \langle x, y\rangle \vee \langle -3, -1\rangle$.
- Then $\mathbf{A}_S$ is the subalgebra of $(\mathbf{G}_{\sigma_S})_\gamma$ generated by $\{a = \langle -1, 0\rangle, b = \langle -1, -1\rangle\}$.

Lemma

The algebra $\mathbf{A}_S$ is simple integral commutative residuated chain generated by a.

Proof.

It is generated by a since

$$b = a^2 \to a^4 = \langle -2, 0\rangle \to \langle -3, -1\rangle = \langle -1, -1\rangle \,.$$

□

Lemma

Let $n \in \mathbb{N}$. Then $\langle -2, n\rangle, \langle -3, n\rangle \in A_S$.

- $\langle -2, n+1\rangle = b \to a\langle -2, n\rangle$ and $\langle -3, n\rangle = a\langle -2, n\rangle$.

Lemma

Let $n \in \mathbb{N}$*. Then* $\langle -2, n\rangle, \langle -3, n\rangle \in A_S$*.*

- $\langle -2, n+1\rangle = b \to a\langle -2, n\rangle$ and $\langle -3, n\rangle = a\langle -2, n\rangle$.

Lemma

Let $z \in S$*. Then* $\langle -1, z\rangle \in A_S$*.*

- We have $S \cup \{-1, 0\} = \{s_0 = 0 > s_1 = -1 > s_2 > \cdots\}$.
- $\langle -1, s_{n+1}\rangle = a \to b\langle -1, s_n\rangle = \langle -1, 0\rangle \to \langle -2, s_n - 1\rangle$.

Lemma

Let $n \in \mathbb{N}$. Then $\langle -2, n\rangle, \langle -3, n\rangle \in A_S$.

- $\langle -2, n+1\rangle = b \to a\langle -2, n\rangle$ and $\langle -3, n\rangle = a\langle -2, n\rangle$.

Lemma

Let $z \in S$. Then $\langle -1, z\rangle \in A_S$.

- We have $S \cup \{-1, 0\} = \{s_0 = 0 > s_1 = -1 > s_2 > \cdots\}$.
- $\langle -1, s_{n+1}\rangle = a \to b\langle -1, s_n\rangle = \langle -1, 0\rangle \to \langle -2, s_n - 1\rangle$.

Lemma

Let $R, S \subseteq -2 - \mathbb{N}$ such that $R \neq S$. Then $\mathbf{A}_R$ is not isomorphic to $\mathbf{A}_S$.

- Finally we extend $\mathbf{A}_S$ by a top element $\top$ such that $\top x = x$ for $x \neq \langle 0, 0 \rangle$. The resulting algebra is denoted $\mathbf{A}_S^{\top}$.

- Finally we extend $\mathbf{A}_S$ by a top element $\top$ such that $\top x = x$ for $x \neq \langle 0, 0 \rangle$. The resulting algebra is denoted $\mathbf{A}_S^\top$.

Lemma

The algebra $\mathbf{A}_S^\top$ is strictly simple with a nearly term definable bottom element by the term $x^4 \wedge (x \to 1)^4$.

- Finally we extend $\mathbf{A}_S$ by a top element $\top$ such that $\top x = x$ for $x \neq \langle 0, 0 \rangle$. The resulting algebra is denoted $\mathbf{A}_S^\top$.

Lemma

The algebra $\mathbf{A}_S^\top$ is strictly simple with a nearly term definable bottom element by the term $x^4 \wedge (x \to 1)^4$.

Theorem (Galatos, Jipsen, Kowalski, Ono)

Let $\mathbf{A}$ be a strictly simple FL-algebra or residuated lattice with bottom element $\bot$ nearly term definable by an n-ary term t that does not involve the constant 0. Then, $\mathsf{V}(\mathbf{A})$ is a minimal variety. Moreover, if $\mathbf{A}'$ is a strictly simple FL-algebra or residuated lattice with bottom element nearly term definable by the same term t, then $\mathsf{V}(\mathbf{A}) \subseteq \mathsf{V}(\mathbf{A}')$ if and only if $\mathbf{A}$ and $\mathbf{A}'$ are isomorphic.

Main results

Theorem

There are $2^{\aleph_0}$ representable commutative 4-potent atoms in $\Lambda(\mathsf{RL})$.

Main results

Theorem

There are $2^{\aleph_0}$ *representable commutative 4-potent atoms in* $\Lambda(\mathsf{RL})$*.*

Theorem

There are only finitely many 3-potent representable commutative atoms. Namely, varieties generated by $\mathbf{2}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_3'$*.*

Theorem

We have

1. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathsf{FL_{ei}})$.

Theorem

We have

1. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathrm{FL_{ei}})$.
2. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathrm{FL_{eo}})$.

Theorem

We have

1. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathsf{FL}_{ei})$.
2. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathsf{FL}_{eo})$.

Proof.

1. We use the FL-algebras living on $\mathbf{A}_S$ where 0 is interpreted by any element different from $\langle -3, -1\rangle, \langle 0, 0\rangle$.

Theorem

We have

1. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathsf{FL}_{ei})$.
2. *There are* $2^{\aleph_0}$ *representable commutative atoms in* $\Lambda(\mathsf{FL}_{eo})$.

Proof.

1. We use the FL-algebras living on $\mathbf{A}_S$ where 0 is interpreted by any element different from $\langle -3, -1\rangle, \langle 0, 0\rangle$.
2. We use the FL-algebras living on $\mathbf{A}_S^\top$ where 0 is interpreted by $\langle -3, -1\rangle$.

□

Representable Commutative Integral Atoms

Theorem

There are 2^1 *representable commutative integral atoms in* $\Lambda(\mathsf{RL})$*, namely* $\mathsf{CLG}^- = \mathsf{V}(\mathbf{Z}^-)$ *and* $\mathsf{GBA} = \mathsf{V}(\mathbf{2})$*.*

Proof.

- Let **A** be a representable simple ICRC.

Proof.

- Let **A** be a representable simple ICRC.
- If **A** has a minimum 0, then $\{0, 1\}$ is a subalgebra isomorphic to **2**.

Proof.

- Let **A** be a representable simple ICRC.
- If **A** has a minimum 0, then $\{0, 1\}$ is a subalgebra isomorphic to **2**.
- If **A** has no minimum, then $\langle a^k \rangle_{k \in \mathbb{N}^+}$ is a strictly decreasing sequence for any element $a \neq 1$.

Proof.

- Let **A** be a representable simple ICRC.
- If **A** has a minimum 0, then $\{0, 1\}$ is a subalgebra isomorphic to **2**.
- If **A** has no minimum, then $\langle a^k \rangle_{k \in \mathbb{N}^+}$ is a strictly decreasing sequence for any element $a \neq 1$.
- Consider a free ultrafilter U on $\mathbb{N}$ and the corresponding ultrapower $\mathbf{B} = \mathbf{A}^{\mathbb{N}}/U$.

Proof.

- Let **A** be a representable simple ICRC.
- If **A** has a minimum 0, then $\{0, 1\}$ is a subalgebra isomorphic to **2**.
- If **A** has no minimum, then $\langle a^k \rangle_{k \in \mathbb{N}^+}$ is a strictly decreasing sequence for any element $a \neq 1$.
- Consider a free ultrafilter U on $\mathbb{N}$ and the corresponding ultrapower $\mathbf{B} = \mathbf{A}^{\mathbb{N}}/U$.
- Let θ be the congruence on **B** corresponding to the convex subalgebra generated by the congruence classes containing the constant mappings and $\mathbf{a} = \langle a^k \rangle_{k \in \mathbb{N}^+}/U$.

Proof.

- Let **A** be a representable simple ICRC.
- If **A** has a minimum 0, then $\{0, 1\}$ is a subalgebra isomorphic to **2**.
- If **A** has no minimum, then $\langle a^k \rangle_{k \in \mathbb{N}^+}$ is a strictly decreasing sequence for any element $a \neq 1$.
- Consider a free ultrafilter U on $\mathbb{N}$ and the corresponding ultrapower $\mathbf{B} = \mathbf{A}^{\mathbb{N}}/U$.
- Let θ be the congruence on **B** corresponding to the convex subalgebra generated by the congruence classes containing the constant mappings and $\mathbf{a} = \langle a^k \rangle_{k \in \mathbb{N}^+}/U$.
- Then the subalgebra of $\mathbf{B}/\theta$ generated by **a** is isomorphic to $\mathbf{Z}^-$, i.e., it belongs to CLG^-.

Proof.

- Let **A** be a representable simple ICRC.
- If **A** has a minimum 0, then $\{0, 1\}$ is a subalgebra isomorphic to **2**.
- If **A** has no minimum, then $\langle a^k \rangle_{k \in \mathbb{N}^+}$ is a strictly decreasing sequence for any element $a \neq 1$.
- Consider a free ultrafilter U on $\mathbb{N}$ and the corresponding ultrapower $\mathbf{B} = \mathbf{A}^{\mathbb{N}}/U$.
- Let θ be the congruence on **B** corresponding to the convex subalgebra generated by the congruence classes containing the constant mappings and $\mathbf{a} = \langle a^k \rangle_{k \in \mathbb{N}^+}/U$.
- Then the subalgebra of $\mathbf{B}/\theta$ generated by **a** is isomorphic to $\mathbf{Z}^-$, i.e., it belongs to CLG$^-$.Let $n > m$. Then

$$a^{(n-m)k} \leq a^{mk} \to a^{nk} < a^{(n-m)k-1} \leq a \to a^{(n-m)k}\,.$$

□

1-generated ICRCs

- The discussed results show that the class of 1-generated representable integral commutative residuated chains (ICRCs) is quite large.

1-generated ICRCs

- The discussed results show that the class of 1-generated representable integral commutative residuated chains (ICRCs) is quite large.
- It turns out that it is sufficiently large to generate the whole variety of representable integral commutative residuated lattices.

1-generated ICRCs

- The discussed results show that the class of 1-generated representable integral commutative residuated chains (ICRCs) is quite large.
- It turns out that it is sufficiently large to generate the whole variety of representable integral commutative residuated lattices.
- Using the conuclear contraction and nuclear retraction, we can prove that each finite ICRC can be embedded into a finite 1-generated ICRC.

1-generated ICRCs

- The discussed results show that the class of 1-generated representable integral commutative residuated chains (ICRCs) is quite large.
- It turns out that it is sufficiently large to generate the whole variety of representable integral commutative residuated lattices.
- Using the conuclear contraction and nuclear retraction, we can prove that each finite ICRC can be embedded into a finite 1-generated ICRC.

Theorem

The variety of representable integral commutative residuated lattices is generated by 1*-generated finite totally ordered members.*

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.
- Let **A** be a finite ICRC such that $A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} = \cdots = 1\}$.

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.
- Let **A** be a finite ICRC such that $A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} = \cdots = 1\}$.
- Let $\mathbf{B} = \mathbf{Z}^- \overset{\rightarrow}{\times} \mathbf{A}$.

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.
- Let **A** be a finite ICRC such that $A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} = \cdots = 1\}$.
- Let $\mathbf{B} = \mathbf{Z}^- \overset{\rightarrow}{\times} \mathbf{A}$.
- We define $B_0 = \{\langle 0, x\rangle \mid x \in A\}$ and $B_k = \{\langle -k, x\rangle \mid x \in A,\ x \leq a_k\}$ for $k \geq 1$.

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.
- Let **A** be a finite ICRC such that $A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} = \cdots = 1\}$.
- Let $\mathbf{B} = \mathbf{Z}^- \overrightarrow{\times} \mathbf{A}$.
- We define $B_0 = \{\langle 0, x\rangle \mid x \in A\}$ and $B_k = \{\langle -k, x\rangle \mid x \in A,\ x \leq a_k\}$ for $k \geq 1$.
- Then $B_\sigma = \bigcup_{k\in\mathbb{N}} B_k$ is a conuclear contraction of **B**.

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.
- Let **A** be a finite ICRC such that $A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} = \cdots = 1\}$.
- Let $\mathbf{B} = \mathbf{Z}^- \overset{\rightarrow}{\times} \mathbf{A}$.
- We define $B_0 = \{\langle 0, x\rangle \mid x \in A\}$ and $B_k = \{\langle -k, x\rangle \mid x \in A,\ x \leq a_k\}$ for $k \geq 1$.
- Then $B_\sigma = \bigcup_{k\in\mathbb{N}} B_k$ is a conuclear contraction of **B**.
- Let γ be the nucleus defined $\gamma(x, y) = \langle x, y\rangle \vee \langle -2(n+1), 1\rangle$.

Sketch of the proof

- The variety of representable ICRLs has FEP, i.e., it is generated by its finite chains.
- Let **A** be a finite ICRC such that $A = \{a_1 < a_2 < \ldots < a_n < a_{n+1} = \cdots = 1\}$.
- Let $\mathbf{B} = \mathbf{Z}^- \overset{\rightarrow}{\times} \mathbf{A}$.
- We define $B_0 = \{\langle 0, x\rangle \mid x \in A\}$ and $B_k = \{\langle -k, x\rangle \mid x \in A,\ x \leq a_k\}$ for $k \geq 1$.
- Then $B_\sigma = \bigcup_{k\in\mathbb{N}} B_k$ is a conuclear contraction of **B**.
- Let γ be the nucleus defined $\gamma(x, y) = \langle x, y\rangle \vee \langle -2(n+1), 1\rangle$.
- Finally, let **C** be the subalgebra of $(\mathbf{B}_\sigma)_\gamma$ generated by $g = \langle -1, a_1\rangle$.

Sketch of the proof (cont.)

Lemma

The 1*-generated ICRC* **C** *contains all* $\langle 0, x\rangle$ *for any* $x \in A$*. Thus* **A** *can be embedded into* **C** *via the mapping* $x \mapsto \langle 0, x\rangle$*.*

Sketch of the proof (cont.)

Lemma

The 1*-generated ICRC* **C** *contains all* $\langle 0, x\rangle$ *for any* $x \in A$*. Thus* **A** *can be embedded into* **C** *via the mapping* $x \mapsto \langle 0, x\rangle$*.*

- We have $g^{2(n+1)} = \langle -2(n+1), 1\rangle$.

Sketch of the proof (cont.)

Lemma

The 1*-generated ICRC* **C** *contains all* $\langle 0, x\rangle$ *for any* $x \in A$*. Thus* **A** *can be embedded into* **C** *via the mapping* $x \mapsto \langle 0, x\rangle$*.*

- We have $g^{2(n+1)} = \langle -2(n+1), 1\rangle$.
- Further, we have $\langle -k, a_k\rangle = g^{2(n+1)-k} \to g^{2(n+1)}$ for $k < 2(n+1)$.

Sketch of the proof (cont.)

Lemma

The 1-generated ICRC **C** *contains all* $\langle 0, x\rangle$ *for any* $x \in A$*. Thus* **A** *can be embedded into* **C** *via the mapping* $x \mapsto \langle 0, x\rangle$*.*

- We have $g^{2(n+1)} = \langle -2(n+1), 1\rangle$.
- Further, we have $\langle -k, a_k\rangle = g^{2(n+1)-k} \to g^{2(n+1)}$ for $k < 2(n+1)$.
- Finally, we have for $k \leq n$:

$$\langle 0, a_k\rangle = \langle -n-1-k, 1\rangle \to \langle -n-1, 1\rangle \cdot \langle -k, a_k\rangle .$$