Generalization of Holland’s Theorem for Residuated Lattices

Rostislav Horčík
joint work with Nikolaos Galatos

Institute of Computer Science
Academy of Sciences of the Czech Republic

Order, Algebra and Logics
Krakow 2011
1. Residuated lattices
1. Residuated lattices

2. Cayley-type theorems
1. Residuated lattices
2. Cayley-type theorems
3. Holland-type theorems
Definition

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map $f : \mathbf{P} \to \mathbf{Q}$ is said to be residuated iff it has a (left) residual $f^\dagger$, i.e.

$$f(x) \leq y \quad \text{iff} \quad x \leq f^\dagger(y).$$
Join-semilattice monoids

Definition

A join-semilattice monoid (s\textsuperscript{\ell}-monoid) is an algebra $M = \langle M, \lor, \cdot, 1 \rangle$, where

- $\langle M, \lor \rangle$ is a semilattice,
- $\langle M, \cdot, 1 \rangle$ is a monoid,
- $a(b \lor c) = ab \lor ac$ and $(b \lor c)a = ba \lor ca$. 

Example

Let $S$ be a join-semilattice. Then $\text{End}(S)$ and $\text{Res}(S)$ are $s\ell$-monoids. Moreover, $\text{Res}(S)$ is a subalgebra of $\text{End}(S)$. 

Rostislav Horčík (ICS)
Definition

A join-semilattice monoid (sℓ-monoid) is an algebra $\mathbf{M} = \langle M, \vee, \cdot, 1 \rangle$, where

- $\langle M, \vee \rangle$ is a semilattice,
- $\langle M, \cdot, 1 \rangle$ is a monoid,
- $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$.

Example

Let $\mathbf{S}$ be a join-semilattice. Then $\text{End}(\mathbf{S})$ and $\text{Res}(\mathbf{S})$ are an sℓ-monoids. Moreover, $\text{Res}(\mathbf{S})$ is a subalgebra of $\text{End}(\mathbf{S})$. 
Residuated lattices

Definition

A residuated lattice is an algebra $\mathbf{A} = \langle A, \land, \lor, \cdot, \div, \backslash, 1 \rangle$, where

- $\langle A, \land, \lor \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid and
- $\cdot$ is residuated component-wise, i.e.

$$\quad x \cdot y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x \backslash z.$$
Residuated lattices

Definition
A **residuated lattice** is an algebra $\mathbf{A} = \langle A, \land, \lor, \cdot, /, \backslash, 1 \rangle$, where

- $\langle A, \land, \lor \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid and
- $\cdot$ is residuated component-wise, i.e.

$$x \cdot y \leq z \text{ iff } x \leq z / y \text{ iff } y \leq x \backslash z.$$ 

Fact
Every residuated lattice forms an $s\ell$-monoid.
**Definition**

A residuated lattice is an algebra $A = \langle A, \land, \lor, \cdot, /, \setminus, 1 \rangle$, where

- $\langle A, \land, \lor \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid and
- $\cdot$ is residuated component-wise, i.e.

$$x \cdot y \leq z \text{ iff } x \leq z / y \text{ iff } y \leq x \setminus z.$$

**Fact**

Every residuated lattice forms an $s\ell$-monoid.

**Example**

Let $L$ be a complete lattice. Then $\text{Res}(L)$ is a complete residuated lattice.
Let $G$ be a group.

\[ G \xrightarrow{\phi} \text{Sym}(G) \]

\[ a \xrightarrow{} a \cdot - \]

What about $\text{Res}(L)$? $L$ need not be complete but it is embeddable into a completion $L$. The image $\phi[L] \subseteq \text{Res}(L)$ but $\phi$ need not preserve $\land$ and $\lor$, $\setminus$. It is only an $s\ell$-monoid embedding.
Let \( S \) be an \( s\ell \)-monoid.

\[
\begin{array}{ccc}
S & \overset{\phi}{\longrightarrow} & \text{End}(S) \\
\downarrow & & \\
a & \longrightarrow & a \cdot -
\end{array}
\]
Cayley-type theorems

- Let $L$ be a residuated lattice.

$$L \xrightarrow{\phi} \text{End}(L)$$

- $\text{End}(L)$ need not be a residuated lattice.

$$a \xrightarrow{\phi} a \cdot -$$
Cayley-type theorems

1. Let $L$ be a residuated lattice.

   $$L \xrightarrow{\phi} \text{Res}(L)$$

   $$a \xrightarrow{} a \cdot -$$

2. $\text{End}(L)$ need not be a residuated lattice.

3. What about $\text{Res}(L)$?
Cayley-type theorems

Let $L$ be a residuated lattice.

$\begin{align*}
L & \xrightarrow{\phi} \overline{L} \\
\text{End}(L) & \xrightarrow{\phi} \overline{L} \\
a & \xrightarrow{\phi} a \cdot \bar{1}
\end{align*}$

- $\text{End}(L)$ need not be a residuated lattice.
- What about $\text{Res}(L)$?
- $L$ need not be complete but it is embeddable into a completion $\overline{L}$. 
Cayley-type theorems

- Let $L$ be a residuated lattice.

$$
\begin{array}{ccc}
L & \xrightarrow{\phi} & \text{Res}(L) \\
\downarrow{\phi} & & \downarrow{\phi} \\
\bar{L} & & \bar{L} \\
\end{array}
$$

- $\text{End}(L)$ need not be a residuated lattice.

- What about $\text{Res}(L)$?

- $L$ need not be complete but it is embeddable into a completion $\bar{L}$.

- The image $\phi[\bar{L}] \subseteq \text{Res}(\bar{L})$ but $\phi$ need not preserve $\wedge$ and $\setminus, \div$. It is only an $s\ell$-monoid embedding.
Conuclei

Definition
A conucleus \( \sigma \) on a residuated lattice \( L \) is an interior operator such that 
\[ \sigma(x)\sigma(y) \leq \sigma(xy) \text{ and } \sigma(1) = 1. \]
Conuclei

Definition
A conucleus $\sigma$ on a residuated lattice $L$ is an interior operator such that $\sigma(x)\sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$.

Theorem
Let $\sigma$ be a conucleus on a residuated lattice $L = \langle L, \land, \lor, \cdot, \backslash, /, 1 \rangle$. Then $\sigma[L] = \langle \sigma[L], \land_{\sigma}, \lor, \cdot, \backslash_{\sigma}, /_{\sigma}, 1 \rangle$ is a residuated lattice, where $x \land_{\sigma} y = \sigma(x \land y)$, $x \backslash_{\sigma} y = \sigma(x \backslash y)$ and $x / y = \sigma(x / y)$.
Conuclei

Definition
A conucleus \( \sigma \) on a residuated lattice \( L \) is an interior operator such that
\[
\sigma(x)\sigma(y) \leq \sigma(xy) \quad \text{and} \quad \sigma(1) = 1.
\]

Theorem
Let \( \sigma \) be a conucleus on a residuated lattice \( L = \langle L, \wedge, \vee, \cdot, \backslash, /, 1 \rangle \).
Then \( \sigma[L] = \langle \sigma[L], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1 \rangle \) is a residuated lattice, where
\[
x \wedge_\sigma y = \sigma(x \wedge y), \quad x \backslash_\sigma y = \sigma(x \backslash y) \quad \text{and} \quad x/y = \sigma(x/y).
\]

Theorem (Montagna-Tsinakis)
Every commutative cancellative residuated lattice is the conucleus image of an Abelian \( \ell \)-group.

Rostislav Horčík (ICS)

Holland’s Theorem

OAL 2.0 7 / 12
Cayley’s theorem for residuated lattices

Theorem

If a residuated lattice $A$ embeds as an $s\ell$-monoid into a complete residuated lattice $B$ (via $f : A \to B$), then it also embeds as a residuated lattice into a conuclear image $\sigma[B]$ of $B$. 
Cayley’s theorem for residuated lattices

**Theorem**

*If a residuated lattice $A$ embeds as an $s\ell$-monoid into a complete residuated lattice $B$ (via $f : A \to B$), then it also embeds as a residuated lattice into a conuclear image $\sigma[B]$ of $B$."

**Proof.**

The map

$$\sigma(x) = \bigvee \{ z \in f[A] \mid z \leq x \}$$

is a conucleus such that $f : A \to \sigma[B]$ preserves $\land, \backslash$ and $\bigvee$. □
Cayley’s theorem for residuated lattices

Theorem

If a residuated lattice $A$ embeds as an $s\ell$-monoid into a complete residuated lattice $B$ (via $f : A \rightarrow B$), then it also embeds as a residuated lattice into a conuclear image $\sigma[B]$ of $B$.

Proof.

The map

$$\sigma(x) = \bigvee \{ z \in f[A] \mid z \leq x \}$$

is a conucleus such that $f : A \rightarrow \sigma[B]$ preserves $\wedge$, $\setminus$ and $/$.

Theorem (Galatos-H.)

Every residuated lattice $L$ embeds into a conuclear image of $\text{Res}(\overline{L})$, where $\overline{L}$ is a completion of $L$ (for example the DM-completion of $L$).
Can we replace $\mathbb{L}$ by a complete chain $\mathbb{C}$?
Holland-type theorems

\[ L \rightarrow \sigma[\text{Res}(\bar{L})] \]

- Can we replace \( \bar{L} \) by a complete chain \( C \)?

**Theorem (Holland)**

*Every \( \ell \)-group can be embedded in the \( \ell \)-group \( \text{Aut}(C) \) of the order-automorphisms on a chain \( C \).*
Holland-type theorems

\[
\begin{array}{ccc}
L & \rightarrow & \sigma[\text{Res}(\overline{L})]
\end{array}
\]

Can we replace \( L \) by a complete chain \( C \)?

**Theorem (Anderson-Edwards)**

*Every distributive \( \ell \)-monoid can be embedded in the \( \ell \)-monoid \( \text{End}(C) \) of the order-preserving maps on a chain \( C \).*

\( \ell \)-monoid is a lattice-ordered monoid where multiplication distributes over meets and joins.
Holland-type theorems

\[ L \rightarrow \sigma[\text{Res}(\overline{L})] \]

Can we replace \( \overline{L} \) by a complete chain \( C \)?

**Theorem (Paoli-Tsinakis)**

*Every distributive residuated lattice in which multiplication distributes over meets can be embedded as \( \ell \)-monoid into \( \text{Res}(C) \) for a complete chain \( C \).*
Holland-type theorems

\[ L \longrightarrow \sigma[\text{Res}(L)] \]

- Can we replace \( L \) by a complete chain \( C \)?

Theorem (Paoli-Tsinakis)

*Every distributive residuated lattice in which multiplication distributes over meets can be embedded as \( \ell \)-monoid into \( \text{Res}(C) \) for a complete chain \( C \).*

- Is it possible that every residuated lattice embeds into a conuclear image of \( \text{Res}(C) \) for a complete chain \( C \)?
Condition (ec)

Let $C$ be a chain. Then $\text{Res}(C)$ is a distributive lattice. But distributivity need not be preserved by a conucleus.
Let $C$ be a chain. Then $\text{Res}(C)$ is a distributive lattice. But distributivity need not be preserved by a conucleus.

$\sigma[L]$ is a subalgebra of $L$ w.r.t. joins, multiplication and 1. So $\sigma[L]$ satisfies the same quasi-identities in the language of s$\ell$-monoids as $L$. 

**Condition (ec)**
Condition (ec)

- Let $\mathbf{C}$ be a chain. Then $\text{Res}(C)$ is a distributive lattice. But distributivity need not be preserved by a conucleus.
- $\sigma[\mathbf{L}]$ is a subalgebra of $\mathbf{L}$ w.r.t. joins, multiplication and 1. So $\sigma[\mathbf{L}]$ satisfies the same quasi-identities in the language of $s\ell$-monoids as $\mathbf{L}$.
- Consider the following quasi-identity:

$$u \leq h \vee zy \quad \& \quad u \leq h \vee wx \implies u \leq h \vee zx \vee wy.$$  
(ec)
Condition (ec)

Let $C$ be a chain. Then $\text{Res}(C)$ is a distributive lattice. But distributivity need not be preserved by a conucleus.

$\sigma[L]$ is a subalgebra of $L$ w.r.t. joins, multiplication and 1. So $\sigma[L]$ satisfies the same quasi-identities in the language of $s\ell$-monoids as $L$.

Consider the following quasi-identity:

$$u \leq h \lor zy \quad \& \quad u \leq h \lor wx \implies u \leq h \lor zx \lor wy. \quad (\text{ec})$$

In the presence of meet, (ec) is equivalent to

$$(h \lor zy) \land (h \lor wx) \leq h \lor zx \lor wy.$$
Condition (ec)

- Let $\mathbf{C}$ be a chain. Then $\text{Res}(C)$ is a distributive lattice. But distributivity need not be preserved by a conucleus.
- $\sigma[\mathbf{L}]$ is a subalgebra of $\mathbf{L}$ w.r.t. joins, multiplication and 1. So $\sigma[\mathbf{L}]$ satisfies the same quasi-identities in the language of $s\ell$-monoids as $\mathbf{L}$.
- Consider the following quasi-identity:
  
  \[ u \leq h \lor zy \quad \& \quad u \leq h \lor wx \quad \implies \quad u \leq h \lor zx \lor wy. \quad \text{(ec)} \]

- In the presence of meet, (ec) is equivalent to
  
  \[ (h \lor zy) \land (h \lor wx) \leq h \lor zx \lor wy. \]

Lemma

Let $\mathbf{C}$ be a complete chain. Then $\text{End}(C)$ satisfies (ec). Thus (ec) holds also in $\text{Res}(C)$ and $\sigma[\text{Res}(C)]$. 
Key lemma

Let \( \mathbf{A} \) be an \( s\ell \)-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism \( f: \mathbf{A} \to \text{End}(\mathbf{C}) \) for some chain \( \mathbf{C} \).
2. \( \mathbf{A} \) has a proper lattice ideal (which can be chosen as maximal w.r.t. not containing an element).

Proof.

Given \( \emptyset \neq H \subseteq \mathbf{A} \) and \( a \in \mathbf{A} \), define \( H/a = \{ c \in \mathbf{A} | ca \in H \} \).

If \( H \) is lattice ideal maximal w.r.t. not containing an element, then \( C = \langle C, \supseteq \rangle \) is a chain, where \( C = \{ H/a | a \in \mathbf{A} \} \).

Then \( f(b) = \phi_b \), where \( \phi_b(H/a) = H/ba \).
Let $A$ be an $s\ell$-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism $f : A \rightarrow \text{End}(C)$ for some chain $C$. 

Proof.

Given $\emptyset \neq H \subseteq A$ and $a \in A$, define $H/a = \{ c \in A \mid ca \in H \}$. If $H$ is lattice ideal maximal w.r.t. not containing an element, then $C = \langle C, \sup \rangle$ is a chain, where $C = \{ H/a \mid a \in A \}$. Then $f(b) = \phi b$, where $\phi b(H/a) = H/ba$. 

Rostislav Horčík (ICS)
Key lemma

Let $A$ be an $s\ell$-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism $f : A \to \text{End}(C)$ for some chain $C$.

2. $A$ has a proper lattice ideal (which can be chosen as maximal w.r.t. not containing an element).
Key lemma

Let $A$ be an $s\ell$-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism $f : A \to \text{End}(C)$ for some chain $C$.

2. $A$ has a proper lattice ideal (which can be chosen as maximal w.r.t. not containing an element).

Proof.
Key lemma

Let $A$ be an $s\ell$-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism $f: A \to \text{End}(C)$ for some chain $C$.

2. $A$ has a proper lattice ideal (which can be chosen as maximal w.r.t. not containing an element).

Proof.

- Given $\emptyset \neq H \subseteq A$ and $a \in A$, define $H/a = \{c \in A \mid ca \in H\}$.
Key lemma

Let $A$ be an $s\ell$-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism $f : A \to \text{End}(C)$ for some chain $C$.
2. $A$ has a proper lattice ideal (which can be chosen as maximal w.r.t. not containing an element).

Proof.

- Given $\emptyset \neq H \subseteq A$ and $a \in A$, define $H/a = \{c \in A \mid ca \in H\}$.
- If $H$ is lattice ideal maximal w.r.t. not containing an element, then $C = \langle C, \supseteq \rangle$ is a chain, where $C = \{H/a \mid a \in A\}$.
Key lemma

Let $A$ be an $s\ell$-monoid satisfying (ec). Then t.f.a.e:

1. There exists a nontrivial homomorphism $f : A \to \text{End}(C)$ for some chain $C$.

2. $A$ has a proper lattice ideal (which can be chosen as maximal w.r.t. not containing an element).

Proof.

- Given $\emptyset \neq H \subseteq A$ and $a \in A$, define $H/a = \{ c \in A \mid ca \in H \}$.
- If $H$ is lattice ideal maximal w.r.t. not containing an element, then $C = \langle C, \supseteq \rangle$ is a chain, where $C = \{ H/a \mid a \in A \}$.
- Then $f(b) = \phi_b$, where $\phi_b(H/a) = H/ba$. 

\[ \square \]
Main theorem

Theorem

For a residuated lattice $A$ the following are equivalent:

1. $A$ satisfies (ec).
2. There exists a complete chain $C$ such that $A$ can be embedded into a conuclear image of $\text{Res}(C)$.
Main theorem

Theorem

For a residuated lattice $A$ the following are equivalent:

1. $A$ satisfies (ec).
2. There exists a complete chain $C$ such that $A$ can be embedded into a conuclear image of $\text{Res}(C)$.

Proof.

Let $\{H_i \mid i \in I\}$ be the collection of all lattice ideals maximal w.r.t. not containing an element. Then we have chains $C_i$ and $s\ell$-monoid homomorphisms $f_i: A \to \text{End}(C_i)$. 


Main theorem

Theorem

For a residuated lattice $A$ the following are equivalent:

1. $A$ satisfies (ec).
2. There exists a complete chain $C$ such that $A$ can be embedded into a conuclear image of $\text{Res}(C)$.

Proof.

Let $\{H_i \mid i \in I\}$ be the collection of all lattice ideals maximal w.r.t. not containing an element. Then we have chains $C_i$ and $s \ell$-monoid homomorphisms $f_i: A \rightarrow \text{End}(C_i)$.

\[
A \rightarrow \prod_{i \in I} \text{End}(C_i) \rightarrow \text{End}(\bigoplus_{i \in I} C_i) \rightarrow \text{Res}(C)
\]

$s \ell$-monoid morphism
**Main theorem**

**Theorem**

For a residuated lattice $A$ the following are equivalent:

1. $A$ satisfies (ec).
2. There exists a complete chain $C$ such that $A$ can be embedded into a conuclear image of $\text{Res}(C)$.

**Proof.**

Let $\{H_i \mid i \in I\}$ be the collection of all lattice ideals maximal w.r.t. not containing an element. Then we have chains $C_i$ and $s\ell$-monoid homomorphisms $f_i: A \to \text{End}(C_i)$.

\[ A \xrightarrow{\text{residuated lattice morphism}} \sigma[\text{Res}(C)] \]