Generalization of Holland's Theorem for Residuated Lattices

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Outline

1. Residuated lattices

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2. Cayley-type theorems

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2. Cayley-type theorems
3. Holland-type theorems

Residuated maps

Definition

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map $f\colon \mathbf{P} \to \mathbf{Q}$ is said to be residuated iff it has a (left) residual $f^{\dagger}$, i.e.

$$f(x) \leq y \qquad \text{iff} \qquad x \leq f^{\dagger}(y)\,.$$

Join-semilattice monoids

Definition

A join-semilattice monoid (sℓ-monoid) is an algebra $\mathbf{M} = \langle M, \vee, \cdot, 1\rangle$, where

- $\langle M, \vee\rangle$ is a semilattice,
- $\langle M, \cdot, 1\rangle$ is a monoid,
- $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$.

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Example

Let **S** be a join-semilattice. Then $\mathbf{End}(S)$ and $\mathbf{Res}(S)$ are an sℓ-monoids. Moreover, $\mathbf{Res}(S)$ is a subalgebra of $\mathbf{End}(S)$.

Residuated lattices

Definition

A residuated lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 1 \rangle$, where

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid and
- $\cdot$ is residuated component-wise, i.e.

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z \,.$$

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Fact

Every residuated lattice forms an sℓ-monoid.

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Fact

Every residuated lattice forms an sℓ-monoid.

Example

Let **L** be a complete lattice. Then $\mathbf{Res}(L)$ is a complete residuated lattice.

Cayley-type theorems

- Let **G** be a group.

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\phi} & \mathbf{Sym}(G) \\ a & \longmapsto & a \cdot - \end{array}$$

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- Let **S** be an sℓ-monoid.

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- $\mathbf{End}(L)$ need not be a residuated lattice.
- What about $\mathbf{Res}(L)$?
- **L** need not be complete but it is embedabble into a completion $\overline{\mathbf{L}}$.
- The image $\phi[\overline{\mathbf{L}}] \subseteq \mathbf{Res}(\overline{L})$ but ϕ need not preserve $\wedge$ and $\backslash, /$. It is only an sℓ-monoid embedding.

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A conucleus σ on a residuated lattice **L** is an interior operator such that $\sigma(x)\sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$.

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Theorem

Let σ be a conucleus on a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$. Then $\sigma[\mathbf{L}] = \langle \sigma[L], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1\rangle$ is a residuated lattice, where $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x/y = \sigma(x/y)$.

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Theorem (Montagna-Tsinakis)

Every commutative cancellative residuated lattice is the conucleus image of an Abelian ℓ-group.

Cayley's theorem for residuated lattices

Theorem

If a residuated lattice $\mathbf{A}$ *embeds as an* $s\ell$*-monoid into a complete residuated lattice* $\mathbf{B}$ *(via* $f\colon A \to B$*), then it also embeds as a residuated lattice into a conuclear image* $\sigma[\mathbf{B}]$ *of* $\mathbf{B}$*.*

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Proof.

The map

$$\sigma(x) = \bigvee \{z \in f[A] \mid z \leq x\}$$

is a conucleus such that $f\colon A \to \sigma[B]$ preserves $\wedge$, $\backslash$ and $/$. □

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Theorem (Galatos-H.)

Every residuated lattice $\mathbf{L}$ *embeds into a conuclear image of* $\mathbf{Res}(\overline{L})$*, where* $\overline{\mathbf{L}}$ *is a completion of* $\mathbf{L}$ *(for example the DM-completion of* $\mathbf{L}$*).*

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$$\mathbf{L} \longrightarrow \sigma[\mathbf{Res}(\overline{L})]$$

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Theorem (Holland)

Every ℓ-group can be embedded in the ℓ-group $\mathbf{Aut}(C)$ of the order-automorphisms on a chain $\mathbf{C}$.

Holland-type theorems

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Theorem (Anderson-Edwards)

Every distributive ℓ-monoid can be embedded in the ℓ-monoid $\mathbf{End}(C)$ of the order-preserving maps on a chain $\mathbf{C}$.

ℓ-monoid is a lattice-ordered monoid where multiplication distributes over meets and joins.

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Theorem (Paoli-Tsinakis)

Every distributive residuated lattice in which multiplication distributes over meets can be embedded as ℓ-monoid into $\mathbf{Res}(C)$ *for a complete chain* $\mathbf{C}$.

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Theorem (Paoli-Tsinakis)

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- Is it possible that every residuated lattice embeds into a conuclear image of $\mathbf{Res}(C)$ for a complete chain $\mathbf{C}$?

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- Let **C** be a chain. Then **Res**(C) is a distributive lattice. But distributivity need not be preserved by a conucleus.

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- $\sigma[\mathbf{L}]$ is a subalgebra of **L** w.r.t. joins, multiplication and 1. So $\sigma[\mathbf{L}]$ satisfies the same quasi-identities in the language of sℓ-monoids as **L**.
- Consider the following quasi-identity:

$$u \leq h \vee zy \quad \& \quad u \leq h \vee wx \quad \Longrightarrow \quad u \leq h \vee zx \vee wy\,. \qquad \text{(ec)}$$

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Lemma

Let **C** *be a complete chain. Then* **End**(C) *satisfies (ec). Thus (ec) holds also in* **Res**(C) *and* $\sigma[\mathbf{Res}(C)]$.

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- Then $f(b) = \phi_b$, where $\phi_b(H/a) = H/ba$.

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Main theorem

Theorem

For a residuated lattice $\mathbf{A}$ *the following are equivalent:*

1. $\mathbf{A}$ *satisfies (ec).*
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Let $\{H_i \mid i \in I\}$ be the collection of all lattice ideals maximal w.r.t. not containing an element. Then we have chains $\mathbf{C}_i$ and sℓ-monoid homomorphisms $f_i \colon \mathbf{A} \to \mathbf{End}(C_i)$.

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$$\mathbf{A} \longrightarrow \prod_{i\in I} \mathbf{End}(C_i) \longrightarrow \mathbf{End}(\bigoplus_{i\in I} C_i) \longrightarrow \mathbf{Res}(C)$$

sℓ-monoid morphism

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$$\mathbf{A} \xrightarrow{\text{residuated lattice morphism}} \sigma[\mathbf{Res}(C)]$$