# On $n$-contractive fuzzy logics

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**Abstract**

It is well known that MTL satisfies the finite embeddability property. Thus MTL is complete w.r.t. the class of all finite MTL-chains. In order to reach a deeper understanding of the structure of this class, we consider the extensions of MTL by adding the generalized contraction since each finite MTL-chain satisfies a form of this generalized contraction. Simultaneously, we also consider extensions of MTL by generalized excludded middle laws introduced in [9] and the axiom of weak cancellation defined in [29]. The algebraic counterpart of these logics is studied characterizing the subdirectly irreducible, the semisimple and the simple algebras. Finally, some important algebraic and logical properties of the considered logics are discussed: local finiteness, finite embeddability property, finite model property, decidability and standard completeness.

**Keywords:** Algebraic Logic, Fuzzy logics, Generalized contraction, Generalized excluded middle, Left-continuous t-norms, MTL-algebras, Non-classical logics, Residuated lattices, Standard completeness, Substructural logics, Varieties, Weak cancellation,

## 1 Introduction

The research on formal systems for fuzzy logic has been growing rapidly during the last years. The origin of this development can be traced back to Hájek's works (see [19]) when he defined the *basic fuzzy logic* BL in order to capture the common fragment of the three main fuzzy logics known at that time: Łukasiewicz logic, Product logic and Gödel logic. These three logics were proved to be standard complete, i.e. complete with respect to the semantics where the set of truth values is the real unit interval $[0, 1]$, the conjunction is interpreted by a continuous t-norm and the implication is interpreted by the residuum of the t-norm. Namely these t-norms were respectively the Łukasiewicz t-norm, the product t-norm and the minimum t-norm, the three main continuous t-norms. In [11] it was proved that BL is, in fact, complete with respect to the semantics given by all continuous t-norms and their residua. Nevertheless, the necessary and sufficient condition for a t-norm to be residuated is not the continuity, but only the left-continuity. For this reason it made perfect sense to consider a more general fuzzy logic system whose semantical completeness would be the class of all left-continuous t-norm and their residua. This logic, MTL, was introduced by Esteva and Godo in [13] and its standard completeness was proved in [23]. Therefore, if we understand fuzzy logic systems as those that are complete with respect to some class of t-norms and their residua, then MTL becomes the weakest fuzzy logic and the research on fuzzy logic systems becomes research on extensions of MTL. Moreover, since it is an algebraizable logic whose equivalent algebraic semantics is the variety of all MTL-algebras, there is a one-to-one correspondence between axiomatic extensions of MTL and subvarieties of MTL-algebras. Some of them are already known (see for instance [25, 15, 12, 16, 17, 30, 31, 33, 29, 20]) but a general description of the structure of all these extensions is still far from being known. In this paper we propose a new way to attack this challenging problem by considering some very general varieties of MTL-algebras, namely the varieties of $n$-contractive MTL-algebras. Some of them were already introduced in [9]. Another reason why to investigate the variety of $n$-contractive MTL-algebras follows from the fact that MTL has the finite embeddability property (as can proved from the results in [5]). This means that MTL is complete w.r.t. the class of all finite MTL-algebras. Thus it is quite natural to study the structure and properties of this class of algebras. One possible approach is to investigate the structure of $n$-contractive MTL-algebras since each finite MTL-algebra is $n$-contractive for some $n \in \mathbb{N}$.

After some necessary general preliminaries in Section 2 about axiomatic extensions of MTL and their algebraization, we consider in Section 3 some equations introduced by Kowalski and Ono in [26] to define $n$-contractive fuzzy logics. Section 4.1 deals with the algebraic counterpart of these logics, the $n$-contractive MTL-algebras; subdirectly irreducible, semisimple and simple algebras are characterized. In Section 4.2 we add some other logics to the hierarchy of $n$-contractive fuzzy logics by means of the weak cancellation law and the $\Omega$ operator and we show that all of them are finitely axiomatizable. Section 4.3 deals with several construction methods with MTL-chains that we need in the sequel. Finally, Section 4.4 is a discussion of some relevant logical and algebraic properties of the considered logics, namely local finiteness, finite embeddability property, finite model property, decidability and standard completeness.[1]

## 2 Preliminaries

In [13] Esteva and Godo define MTL (Monoidal T-norm based Logic) as the sentential logic in the language $\mathcal{L} = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0\rangle$ given by a Hilbert-style calculus with the inference rule of *Modus Ponens* and the following axioms (using implication as the least binding connective):

| | |
|---|---|
| (A1) | $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| (A2) | $\varphi\&\psi \to \varphi$ |
| (A3) | $\varphi\&\psi \to \psi\&\varphi$ |
| (A4) | $\varphi \wedge \psi \to \varphi$ |
| (A5) | $\varphi \wedge \psi \to \psi \wedge \varphi$ |
| (A6) | $\varphi\&(\varphi \to \psi) \to \varphi \wedge \psi$ |
| (A7a) | $(\varphi \to (\psi \to \chi)) \to (\varphi\&\psi \to \chi)$ |
| (A7b) | $(\varphi\&\psi \to \chi) \to (\varphi \to (\psi \to \chi))$ |
| (A8) | $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ |
| (A9) | $\overline{0} \to \varphi$ |

Other usual connectives are defined by:

$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$;

$\varphi \leftrightarrow \psi := (\varphi \to \psi)\&(\psi \to \varphi)$;

$\neg\varphi := \varphi \to \overline{0}$;

$\overline{1} := \neg\overline{0}$.

We denote by $Fm_{\mathcal{L}}$ the set of $\mathcal{L}$-formulae (built from a countable set of variables). If $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we write $\Gamma \vdash_{\mathrm{MTL}} \varphi$ if, and only if, $\varphi$ is derivable from $\Gamma$ in the given calculus. We write $\vdash_{\mathrm{MTL}} \varphi$ instead of $\emptyset \vdash_{\mathrm{MTL}} \varphi$.

**Definition 1.** *A bounded integral commutative residuated lattice is an algebra* $\mathcal{A} = \left\langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \right\rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0\rangle$ *such that:*

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ *is a bounded lattice.*
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ *is a commutative monoid.*
3. $\to^{\mathcal{A}}$ *is the residuum of* $\&^{\mathcal{A}}$*: for every* $a, b, c \in A$*,* $a\&^{\mathcal{A}}b \leq c$ *iff* $a \leq b \to^{\mathcal{A}} c$.

[1] A preliminar presentation of some results included in this paper is available in [32].

We will often omit the supperscripts in the operations of the algebras when they are clear from the context.

**Definition 2** ([13])**.** *An* MTL*-algebra is a bounded integral commutative residuated lattice satisfying the prelinearity equation:*

$$(x \to y) \vee (y \to x) \approx \overline{1}$$

*The negation operation is defined as* $\neg^{\mathcal{A}} a = a \to^{\mathcal{A}} \overline{0}^{\mathcal{A}}$*. If the lattice order is total we will say that* $\mathcal{A}$ *is an* MTL*-chain. The* MTL*-chains defined over the real unit interval* $[0,1]$ *(with the usual order) are those where* $\&^{\mathcal{A}}$ *is a left-continuous t-norm*[2] *and they are called standard* MTL*-chains. If* $\circ$ *is a left-continuous t-norm,* $[0,1]_\circ$ *will denote the standard chain given by* $\circ$*.*

It is well known that the class of all MTL-algebras is a variety. We will denote it as $\mathbb{MTL}$.

**Definition 3.** *Given a class* $\mathbb{K}$ *of* MTL*-algebras and a set of formulae* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*, we define:*

$\Gamma \models_{\mathbb{K}} \varphi$ *iff for all* $\mathcal{A} \in \mathbb{K}$ *and for all evaluation* $v$ *in* $\mathcal{A}$*,* $v[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$ *implies* $v(\varphi) = \overline{1}^{\mathcal{A}}$*.*

*If* $\mathbb{K} = \{\mathcal{A}\}$*, then we write* $\Gamma \models_{\mathcal{A}} \varphi$ *instead of* $\Gamma \models_{\{\mathcal{A}\}} \varphi$*.*

Then, one can prove this theorem of strong completeness for MTL logic:

**Theorem 4.** *If* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*, then* $\Gamma \models_{\mathbb{MTL}} \varphi$ *iff* $\Gamma \vdash_{\mathrm{MTL}} \varphi$*.*

But in fact the relation between MTL and the variety $\mathbb{MTL}$ is much stronger since MTL is an algebraizable logic in the sense of Blok and Pigozzi (see [4]) whose equivalent algebraic semantics is the variety $\mathbb{MTL}$. So all the axiomatic extensions of MTL are also algebraizable in this sense and there is a dual order isomorphism between axiomatic extensions of MTL and subvarieties of $\mathbb{MTL}$, using a translation of formulae into equations and viceversa.

1. $\Gamma \subseteq Fm_{\mathcal{L}}$, $\mathrm{L} = \mathrm{MTL} + \Gamma$. Then the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ axiomatized by the equations $\{\varphi \approx \overline{1} : \varphi \in \Gamma\}$. We denote this variety by $\mathbb{L}$ and we call its members *L-algebras.*
2. $\mathbb{L} \subseteq \mathbb{MTL}$ subvariety axiomatized by a set of equations $\Sigma$. Then the logic associated to $\mathbb{L}$ is the axiomatic extension L of MTL given by the axiom schemata $\{\varphi \leftrightarrow \psi : \varphi \approx \psi \in \Sigma\}$.

[2]A t-norm is a binary operation $\circ : [0,1]^2 \to [0,1]$ which is associative, commutative, isotonic and has 1 as a neutral element (see [24]).

| Axiom schema | Name |
|---|---|
| $\neg\neg\varphi \to \varphi$ | (Inv) |
| $\neg\varphi \vee ((\varphi \to \varphi\&\psi) \to \psi)$ | (C) |
| $\neg(\varphi\&\psi) \vee ((\varphi \to \varphi\&\psi) \to \psi)$ | (WC) |
| $\varphi \to \varphi\&\varphi$ | ($C_2$) |
| $\varphi \wedge \psi \to \varphi\&(\varphi \to \psi)$ | (Div) |
| $\varphi \wedge \neg\varphi \to \overline{0}$ | (PC) |
| $(\varphi\&\psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi\&\psi)$ | (WNM) |

Table 1: Several axiom schemata used in the framework of fuzzy logics.

In the study of these subvarieties the chains play a crucial role due to the next results:

**Theorem 5** ([13]). *Each* MTL-*algebra is isomorphic to a subdirect product of* MTL-*chains.*

**Corollary 6** ([13]). *For every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*,* $\Gamma \vdash_{\mathrm{MTL}} \varphi$ *iff* $\Gamma \models_{\mathcal{A}} \varphi$ *for every* MTL-*chain* $\mathcal{A}$*.*

The same kind of result is true for every axiomatic extension of MTL. It is also possible to restrict the semantics to the algebras defined in the real unit interval by a left-continuous t-norm and its residuum, obtaining the so-called standard completeness results. If a logic L is an axiomatic extension of MTL, we say that L enjoys *(finite) strong standard completeness* if, and only if, for every (finite) set of formulae $T \subseteq Fm_{\mathcal{L}}$ and every formula $\varphi$, $T \vdash_L \varphi$ iff $T \models_{\mathcal{A}} \varphi$ for every standard L-algebra $\mathcal{A}$. We will call this property (F)SSC, for short. We say that L enjoys the *standard completeness* (SC, for short) if, and only if, the equivalence is true for $T = \emptyset$.

Tables 1 and 2 collect some axiom schemata and important axiomatic extensions of MTL that are defined by adding them to the Hilbert-style calculus given above for MTL.

The $\overline{0}$-free subreducts of MTL-algebras are called *prelinear semihoops* and they are defined as follows:

**Definition 7** ([14]). *An algebra* $\mathcal{A} = \left\langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \right\rangle$ *of type* $\langle 2,2,2,0 \rangle$ *is a prelinear semihoop*[3] *iff:*

- $\mathcal{A} = \left\langle A, \wedge^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \right\rangle$ *is an inf-semilattice with upper bound.*

[3]These algebras are sometimes also called MTLH-algebras.

| Logic | Axiom schemata |
| --- | --- |
| SMTL | (PC) |
| WCMTL | (WC) |
| ΠMTL | (C) |
| IMTL | (Inv) |
| WNM | (WNM) |
| NM | (Inv) and (WNM) |
| BL | (Div) |
| SBL | (Div) and (PC) |
| Ł | (Div) and (Inv) |
| Π | (Div) and (C) |
| G | $(C_2)$ |

Table 2: Some axiomatic extensions of MTL and their defining axioms.

- $\left\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \right\rangle$ *is a commutative monoid isotonic with respect to the inf-semilattice order.*
- *For every* $a, b \in A$, $a \leq b$ *iff* $a \rightarrow^{\mathcal{A}} b = \overline{1}^{\mathcal{A}}$.
- *For every* $a, b, c \in A$, $a \&^{\mathcal{A}} b \rightarrow^{\mathcal{A}} c = a \rightarrow^{\mathcal{A}} (b \rightarrow^{\mathcal{A}} c)$.
- *For every* $a, b, c \in A$, $(a \rightarrow^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} c \leq ((b \rightarrow^{\mathcal{A}} a) \rightarrow^{\mathcal{A}} c) \rightarrow^{\mathcal{A}} c$.

*An operation* $\vee^{\mathcal{A}}$ *is defined as:* $a \vee^{\mathcal{A}} b = ((a \rightarrow^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} b) \wedge^{\mathcal{A}} ((b \rightarrow^{\mathcal{A}} a) \rightarrow^{\mathcal{A}} a)$. *If in addition it has a minimum element, then it is a bounded prelinear semihoop (i.e. an* MTL*-algebra).*

**Definition 8** ([2]). *Let* $\langle I, \leq \rangle$ *be a totally ordered set. Let* $\{\mathcal{A}_i \mid i \in I\}$ *be a family of totally ordered semihoops sharing the same top element, say* $\overline{1}$, *and such that for* $i \neq j$, $A_i \cap A_j = \{\overline{1}\}$. *Then* $\bigoplus_{i \in I} \mathcal{A}_i$ *(the ordinal sum of the family) is the totally ordered semihoop whose universe is* $\bigcup_{i \in I} A_i$ *and whose operations are:*

$$x \& y = \begin{cases} x \&^{\mathcal{A}_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{\overline{1}\} \text{ with } i > j, \\ x & \text{if } x \in A_i \setminus \{\overline{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j, \\ \overline{1} & \text{if } x \in A_i \setminus \{\overline{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

*For every $i \in I$, $\mathcal{A}_i$ is called a* component *of the ordinal sum.*

*If in addition $I$ has a minimum, say $i_0$, and $\mathcal{A}_{i_0}$ is bounded, then the ordinal sum $\bigoplus_{i \in I} \mathcal{A}_i$ forms an* MTL*-chain.*

**Definition 9.** *Let $\mathcal{A}$ be an* MTL*-chain or a totally ordered semihoop. We define a binary relation $\sim$ on $A$ by letting for every $a, b \in A$, $a \sim b$ if, and only if, there is $n \geq 1$ such that $a^n \leq b \leq a$ or $b^n \leq a \leq b$. It is easy to check that $\sim$ is an equivalence relation. Its equivalence classes are called* Archimedean classes*. Given $a \in A$, its Archimedean class is denoted as $[a]_\sim$.*

**Definition 10.** *A totally ordered semihoop is indecomposable if, and only if, it is not isomorphic to any ordinal sum of two non-trivial totally ordered semihoops.*

**Theorem 11** ([29])**.** *For every* MTL*-chain $\mathcal{A}$, there is the maximum decomposition as ordinal sum of indecomposable totally ordered semihoops, with the first one bounded.*

**Corollary 12** ([29])**.** *Let $\mathcal{A}$ be an* MTL*-chain. If the partition $\{[a]_\sim : a \in A \setminus \{\overline{1}^{\mathcal{A}}\}\}$ given by the Archimedean classes gives a decomposition as ordinal sum, then it is the maximum one. In this case we say that $\mathcal{A}$ is* totally decomposable.

**Proposition 13.** *Let $\mathcal{A}$ be an* MTL*-chain. The following are equivalent:*

- *$\mathcal{A}$ is totally decomposable.*
- *If $[a]_\sim \neq [b]_\sim$, then $a \& b = a \wedge b$.*
- *For every $a \in A$, $[a]_\sim \cup \{\overline{1}^{\mathcal{A}}\}$ is closed under $\rightarrow$.*

*Proof:* First, we will prove that the second and the third statement are equivalent. Suppose that $[a]_\sim \neq [b]_\sim$ implies $a \& b = a \wedge b$ and take any $x, y \in [a]_\sim$. If $x \leq y$ then $x \rightarrow y = \overline{1} \in [a]_\sim \cup \{\overline{1}^{\mathcal{A}}\}$. Assume $x > y$ and $x \rightarrow y \notin [a]_\sim$. Then $x \& (x \rightarrow y) = x \wedge (x \rightarrow y) = x$ by our assumption. But $x \& (x \rightarrow y) \leq y$ in any MTL-algebra (a contradiction).

Now assume that each $[a]_\sim \cup \{\overline{1}^{\mathcal{A}}\}$ is closed under $\rightarrow$. Take any $x \in [a]_\sim$ and $y \in [b]_\sim$ such that $[a]_\sim \neq [b]_\sim$. Without any loss of generality suppose that $x < y$. Then $x \& y \in [a]_\sim$. Since $x \rightarrow x \& y \geq y$ and $[a]_\sim \cup \{\overline{1}^{\mathcal{A}}\}$ is closed under $\rightarrow$, we get $x \rightarrow x \& y = \overline{1}^{\mathcal{A}}$. Thus $x = x \& y = x \wedge y$.

Finally, it is clear that the second statement together with the third one are equivalent to the claim that $\mathcal{A}$ is totally decomposable. □

Now we recall the operator $\Omega$ (introduced in [29]) acting on varieties of MTL-algebras which closes a given variety under the ordinal sum.

**Definition 14.** *Let* L *be an axiomatic extension of* MTL*. We define* $\Omega(\mathbb{L})$ *as the variety of* MTL*-algebras generated by all the ordinal sums of* $\overline{0}$*-free subreducts of L-chains with the first bounded, and we denote by* $\Omega(L)$ *its corresponding logic.*

Some well known subvarieties of $\mathbb{MTL}$ are closed under this operator, for instance:

- $\Omega(\mathbb{G}) = \mathbb{G}$
- $\Omega(\mathbb{BL}) = \mathbb{BL}$
- $\Omega(\mathbb{SBL}) = \mathbb{SBL}$
- $\Omega(\mathbb{SMTL}) = \mathbb{SMTL}$
- $\Omega(\mathbb{MTL}) = \mathbb{MTL}$

In some other cases they are not closed but we obtain an already known variety:

- $\Omega(\mathbb{BA}) = \mathbb{G}$ ($\mathbb{BA}$ denotes the variety of Boolean algebras)
- $\Omega(\mathbb{MV}) = \mathbb{BL}$ ($\mathbb{MV}$ denotes the variety of MV-algebras)

A filter in an MTL-algebra $\mathcal{A}$ is any subset $F \subseteq A$ such that:

- $\overline{1}^{\mathcal{A}} \in F$
- If $a \in F$ and $a \leq b$, then $b \in F$
- If $a, b \in F$, then $a \& b \in F$.

$F(a)$ will denote the principal filter generated by the element $a$. It can be described as follows: $F(a) = \{b : a^n \leq b \text{ for some } n \geq 1\}$. There is the usual correspondence between filters and congruences in MTL-algebras:

**Proposition 15.** *Let* $\mathcal{A}$ *be an* MTL*-algebra. For every filter* $F \subseteq A$ *we define* $\Theta(F) := \{\langle a, b\rangle \in A^2 : a \leftrightarrow b \in F\}$*, and for every congruence* $\theta$ *of* $\mathcal{A}$ *we define* $Fi(\theta) := \{a \in A : \langle a, \overline{1}^{\mathcal{A}}\rangle \in \theta\}$*. Then,* $\Theta$ *is an order isomorphism from the set of filters onto the set of congruences and* $Fi$ *is its inverse.*

Given a filter $F$ and an element $a$, $[a]_F$ will denote the equivalence class of $a$ w.r.t. to the congruence $\Theta(F)$.

We also need to recall some relevant properties from Universal Algebra.

**Definition 16.** *A class $\mathbb{K}$ of algebras is locally finite (LF, for short) if, and only if, for every $\mathcal{A} \in \mathbb{K}$ and for every finite set $B \subseteq A$, the subalgebra generated by $B$, $\langle B \rangle_{\mathcal{A}}$, is also finite.*

**Definition 17.** *Let $\mathcal{A} = \langle A, \langle f_i : i \in I \rangle \rangle$ be an algebra and let $B \subseteq A$ be an non-empty set. The partial subalgebra $\mathcal{B}$ of $\mathcal{A}$ with domain $B$ is the partial algebra $\langle B, \langle f_i : i \in I \rangle \rangle$, where for every $i \in I$, $f_i$ $n$-ary, $b_1, \ldots, b_n \in B$,*

$$f_i^{\mathcal{B}}(b_1, \ldots, b_n) = \begin{cases} f_i^{\mathcal{A}}(b_1, \ldots, b_n) & \text{if } f_i^{\mathcal{A}}(b_1, \ldots, b_n) \in B, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Given a class $\mathbb{K}$ of algebras, $\mathbb{K}_{fin}$ will denote the class of its finite members.

**Definition 18.** *A class $\mathbb{K}$ of algebras has the finite embeddability property (FEP, for short) if, and only if, every finite partial subalgebra of some member of $\mathbb{K}$ can be embedded in some algebra of $\mathbb{K}_{fin}$.*

**Definition 19.** *A class $\mathbb{K}$ of algebras of the same type has the strong finite model property (SFMP, for short) if, and only if, every quasiequation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.*

**Definition 20.** *A class $\mathbb{K}$ of algebras of the same type has the finite model property (FMP, for short) if, and only if, every equation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.*

A variety has the FMP if, and only if, it is generated by its finite members and a quasivariety has the SFMP if, and only if, it is generated (as a quasivariety) by its finite members. In [5] it is proved that for classes of algebras of finite type closed under finite products (hence, in particular, for varieties of MTL-algebras) the FEP and the SFMP are equivalent. Moreover, it is clear that for every class of algebras $\mathbb{L}$ which is the equivalent algebraic semantics of a logic L, we have:

- If $\mathbb{L}$ is locally finite, then it has the FEP.
- If $\mathbb{L}$ has the FEP, then it has the FMP.
- If $\mathbb{L}$ has the FMP, then L is decidable.

None of these implications can be inverted in general.

The Table 3 shows which of the mentioned properties hold for the axiomatic extensions of MTL in Table 2.

| | LF | FEP = SFMP | FMP | Decidable | SC | FSSC | SSC |
|---|---|---|---|---|---|---|---|
| MTL | No | Yes | Yes | Yes | Yes | Yes | Yes |
| IMTL | No | Yes | Yes | Yes | Yes | Yes | Yes |
| SMTL | No | Yes | Yes | Yes | Yes | Yes | Yes |
| ΠMTL | No | No | No | Yes | Yes | Yes | No |
| BL | No | Yes | Yes | Yes | Yes | Yes | No |
| SBL | No | Yes | Yes | Yes | Yes | Yes | No |
| Π | No | No | No | Yes | Yes | Yes | No |
| G | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| Ł | No | Yes | Yes | Yes | Yes | Yes | No |
| WNM | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| NM | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| CPC | Yes | Yes | Yes | Yes | No | No | No |

Table 3: Algebraic and logical properties for some axiomatic extensions of MTL.

# 3 The $n$-contraction

In [26] Kowalski and Ono studied some varieties of bounded integral commutative residuated lattices. In particular, they considered for every $n \geq 2$ the varieties defined by the following equations:

$$\begin{aligned} (E_n) &\quad x^n \approx x^{n-1} \\ (EM_n) &\quad x \vee \neg x^{n-1} \approx \overline{1} \end{aligned}$$

$(E_2)$ corresponds, in fact, to the law of contraction, which defines the variety of Heyting algebras. Therefore, for every $n \geq 3$ the equation $(E_n)$ corresponds to a weaker form of contraction that we will call *$n$-contraction*. Notice that $(EM_2)$ is the algebraic form of the excluded middle law, and for every $n \geq 3$ $(EM_n)$ corresponds to a weaker form of this law.

In [9] Ciabattoni, Esteva and Godo brought the equations $(E_n)$ to the framework of fuzzy logics. Indeed, for each $n \geq 2$, they defined the $n$-contraction axiom as:

$$\varphi^{n-1} \to \varphi^n \qquad (C_n)$$

and they called $\mathrm{C}_n$MTL (resp. $\mathrm{C}_n$IMTL) the extension of MTL (resp. IMTL) obtained by adding this axiom.

Given $n \geq 2$, the equivalent algebraic semantics of $\mathrm{C}_n$MTL (resp. $\mathrm{C}_n$IMTL) is the class of $n$-contractive MTL-algebras (resp. IMTL-algebras), i.e. the subvariety of $\mathbb{MTL}$ (resp. $\mathbb{IMTL}$) defined by the equation:

$$x^{n-1} \approx x^n$$

Strong standard completeness for these logics was also proved in [9]:

**Theorem 21** ([9]). *For every* $n \geq 3$, $\mathrm{C}_n$MTL *and* $\mathrm{C}_n$IMTL *enjoy the SSC.*

It is easy to see that $\mathrm{C}_2$MTL is Gödel logic and $\mathrm{C}_2$IMTL is the classical propositional calculus. Moreover, for every $n \geq 3$, WNM is a strict extension of $\mathrm{C}_n$MTL, NM is a strict extension of $\mathrm{C}_n$IMTL, $\mathrm{C}_n$MTL is a strict extension of $\mathrm{C}_{n+1}$MTL and $\mathrm{C}_n$IMTL is a strict extension of $\mathrm{C}_{n+1}$IMTL, as depicted in Figure 1.

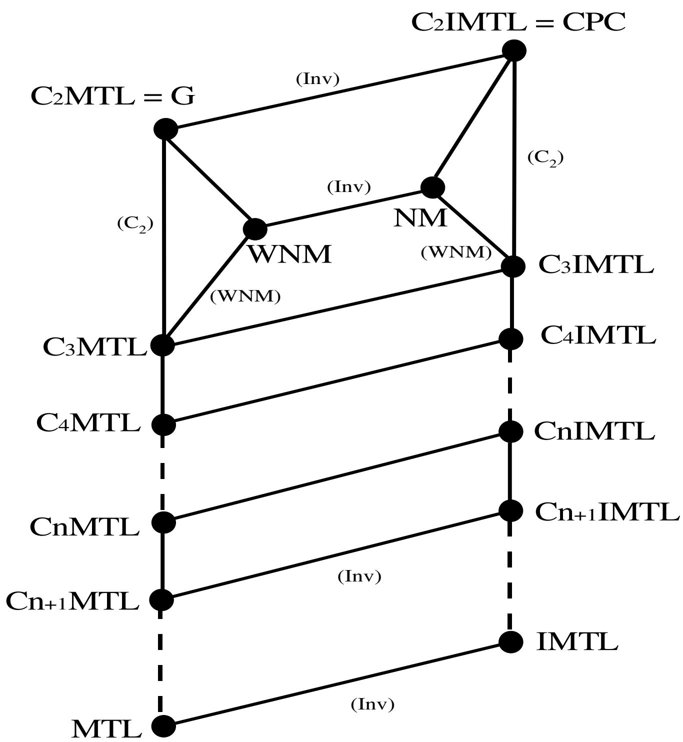


Figure 1: Graphic of axiomatic extensions of MTL obtained by adding all combinations of the schemata (Inv), ($\mathrm{C}_n$) and (WNM). All the depicted inclusions are proper.

We say that an axiomatic extension L of MTL is $n$-contractive if $\vdash_{\mathrm{L}} (C_n)$. Of course, given any L we can make it $n$-contractive by adding the schema

$(C_n)$. We call the resulting logic $\mathrm{C}_n\mathrm{L}$.

In [26] Kowalski and Ono prove the following result:

**Proposition 22** (Prop 1.11, [26])**.** *Let $\mathbb{K}$ be a variety of residuated lattices. Then, $\mathbb{K}$ has the property EDPC (equationally definable principal congruences) if, and only if, $\mathbb{K} \models (E_n)$, for some $n \geq 2$.*

According to a bridge theorem of Abstract Algebraic Logic, an algebraizable logic has the global deduction-detachment theorem if, and only if, its equivalent algebraic semantics has the EDPC. Therefore, in our framework of fuzzy logics as axiomatic extensions of MTL, the contractive logics are a good choice in the sense that they are the only finitary extensions of MTL enjoying the global deduction-detachment theorem.

**Theorem 23.** *If* L *is an $n$-contractive axiomatic extension of* MTL*, then for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$ we have:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \textit{ if, and only if, } \Gamma \vdash_{\mathrm{L}} \varphi^{n-1} \rightarrow \psi.$$

We will consider also the axioms corresponding to $(EM_n)$:

$$\varphi \vee \neg\varphi^{n-1} \qquad (S_n)$$

Given any axiomatic extension L of MTL, $\mathrm{S}_n\mathrm{L}$ will be its extension with $(S_n)$.

# 4 New results

## 4.1 $n$-contractive chains

In this section we will study some basic properties of the $n$-contractive chains. First, observe that this is an important and big class of chains since it contains all the finite MTL-chains.

**Proposition 24.** *All finite* MTL*-chains are $n$-contractive for some $n$.*

*Proof:* Let $\mathcal{A}$ be a finite MTL-chain with $n$ elements. Take an arbitrary $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$. For every $i > 0$, $a^i \leq a^{i-1}$, thus necessarily $a^{n-1} = a^n$. □

**Definition 25.** *Given an* MTL*-algebra $\mathcal{A}$, $a \in A$ is idempotent iff $a^2 = a$. $Id(\mathcal{A})$ will be the set of all idempotent elements of $\mathcal{A}$. Notice that $\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \in Id(\mathcal{A})$.*

**Proposition 26.** *Let $\mathcal{A}$ be an* MTL*-algebra and $a \in Id(\mathcal{A})$. Then for every $b, c \in A$,*

*(1) If $b, c \geq a$, then $b \& c \geq a$.*

*(2) If $b \geq a$, then $a\&b = a$.*

*Proof:* If $b, c \geq a$, then $a = a\&a \leq b\&c$. If $b \geq a$, then $a = a\&a \leq a\&b$, and the other inequality is always true. □

The idempotent elements are easily described in $n$-contractive chains and, in addition, their number can be expressed equationally as the following propositions show.

**Proposition 27.** *Let $\mathcal{A}$ be an $n$-contractive* MTL*-algebra. Then, $Id(\mathcal{A}) = \{a^{n-1} : a \in A\}$.*

*Proof:* If $a \in A$ is idempotent, then $a = a^2 = \ldots = a^{n-1}$. Conversely, take any $a \in A$ and consider $a^{n-1}$. Then, $a^{n-1}\&a^{n-1} = a^n\&a^{n-2} = a^{n-1}\&a^{n-2} = \ldots = a^{n-1}$, so $a^{n-1} \in Id(\mathcal{A})$. □

**Definition 28.** *For every $n \geq 3$ and $k \geq 2$, we define the next formula:*

$I^n_k(x_0, \ldots, x_k) := \bigvee_{i<k}(x_i^{n-1} \to x_{i+1}^{n-1})$.

**Proposition 29.** *For every $n \geq 3$, every $k \geq 2$ and every $n$-contractive* MTL*-chain $\mathcal{A}$ the following are equivalent:*

*(1) $\mathcal{A} \models I^n_k(x_0, \ldots, x_k) \approx \overline{1}$.*

*(2) $\mid Id(\mathcal{A}) \mid \leq k$.*

*Proof:* Suppose that $\mid Id(\mathcal{A}) \mid > k$. Then we can take $a_0, \ldots, a_k \in Id(\mathcal{A})$ such that $a_0 > a_1 > \ldots > a_k$. Then for every $i$, $a_i = a_i^{n-1}$ and $a_{i+1} = a_{i+1}^{n-1}$, so $a_i^{n-1} \to a_{i+1}^{n-1} \neq \overline{1}^{\mathcal{A}}$ and the equation is not satisfied. Conversely, suppose $\mid Id(\mathcal{A}) \mid \leq k$ and take arbitrary elements $a_0, \ldots, a_k \in A$. Since $a_0^{n-1}, \ldots, a_k^{n-1} \in Id(\mathcal{A})$, there are $i < j \leq k$ such that $a_i^{n-1} = a_j^{n-1}$. Hence there is an $l < k$ such that $a_l^{n-1} \to a_{l+1}^{n-1} = \overline{1}^{\mathcal{A}}$. □

In $n$-contractive chains we can also give a nice description of Archimedean classes:

**Proposition 30.** *Let $\mathcal{A}$ be an $n$-contractive* MTL*-chain. Then, for every $a, b \in A$:*

*(i) $a \sim b$ if, and only if, $a^{n-1} = b^{n-1}$, and*

*(ii) $a^{n-1} = \min[a]_\sim$.*

*Proof:*

(i) One direction is obvious. For the other one, suppose that $a \sim b$ and, for instance, $a \leq b$. Then $a^{n-1} \leq b^{n-1}$. Further, there is $i \geq 1$ such that $b^i \leq a \leq b$, hence by the $n$-contraction law $b^{n-1} \leq a \leq b$. Since $b^{n-1}$ is an idempotent smaller than $a$ using Proposition 26 we obtain $b^{n-1} \leq a^{n-1}$.

(ii) It is clear that $a^{n-1} \in [a]_\sim$. Take an arbitrary $b \in [a]_\sim$. By (i), $a^{n-1} = b^{n-1}$, hence $b \geq b^{n-1} = a^{n-1}$.

□

**Corollary 31.** *Let $\mathcal{A}$ be an $n$-contractive* MTL*-chain and let $a \in A$. If $[a]_\sim$ has supremum, then it is the maximum.*

*Proof:* Assume that $b$ is the supremum of $[a]_\sim$. Then, $b^{n-1} = (\sup\{x \in A \mid x^{n-1} = a^{n-1}\})^{n-1} = \sup\{x^{n-1} \in A \mid x^{n-1} = a^{n-1}\} = a^{n-1}$, hence $b \in [a]_\sim$. □

Therefore, Archimedean classes with supremum in $n$-contractive chains are always intervals of the form $[b^{n-1}, b]$. Moreover, this implies that given a standard $n$-contractive chain $\mathcal{A}$, in the set $Id(\mathcal{A})$ none of the elements has neither predecessor nor successor. In particular, 1 is an accumulation point of idempotent elements.[4]

Next proposition characterizes the subdirectly irreducible $n$-contractive algebras.

**Proposition 32.** *Let $\mathcal{A}$ be an $n$-contractive* MTL*-chain. Then:*

*$\mathcal{A}$ is subdirectly irreducible if, and only if, the set of idempotent elements has a coatom.*

*Proof:* First suppose that $\mathcal{A}$ is subdirectly irreducible and let $F$ be the minimum non-trivial filter. Given any $a \in F \setminus \{\overline{1}^{\mathcal{A}}\}$, it is clear that $a^{n-1}$ is a coatom of $Id(\mathcal{A})$. Conversely, suppose $a$ is the coatom in the set of idempotent elements. Then for every $b$ such that $a < b < \overline{1}^{\mathcal{A}}$, we have $b^{n-1} = a$, so $[a, \overline{1}^{\mathcal{A}}]$ is the least non-trivial filter and $\mathcal{A}$ is subdirectly irreducible. □

**Corollary 33.** *There are no subdirectly irreducible standard $n$-contractive* MTL*-chains.*

Nevertheless, notice that this does not contradict the fact that the varieties $\mathbb{C}_n\mathbb{MTL}$ and $\mathbb{C}_n\mathbb{IMTL}$ are generated by their standard chains.

An important subclass of subdirectly irreducible algebras is the class of simple algebras, those without non-trivial congruences. They admit the following general characterization.

[4]These remarks on $n$-contractive left-continuous t-norms are already available in [28].

**Proposition 34.** *Let $\mathcal{A}$ be an* MTL*-algebra. $\mathcal{A}$ is simple if, and only if, for every $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$, there is $k \geq 1$ such that $a^k = \overline{0}^{\mathcal{A}}$.*

The generalized excluded middle equations $(EM_n)$ describe exactly the simple $n$-contractive chains.

**Proposition 35.** *Let $\mathcal{A}$ be an* MTL*-chain. The following are equivalent:*

*(i)* $\mathcal{A} \models (EM_n)$.

*(ii)* $\mathcal{A}$ *is $n$-contractive and simple.*

*Proof:* $(i) \Rightarrow (ii)$ : If $\mathcal{A} \models (EM_n)$, then for every $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$, $a^{n-1} = \overline{0}^{\mathcal{A}}$. Therefore, for every $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$, $a^{n-1} = a^n$ and, by the last proposition, the chain simple. $(ii) \Rightarrow (i)$ : If $\mathcal{A}$ is $n$-contractive and simple, then for $a \in A$, $a^{n-1}$ is idempotent. Therefore, if $a \neq \overline{1}^{\mathcal{A}}$, then $a^{n-1} = \overline{0}^{\mathcal{A}}$, and hence $\mathcal{A} \models (EM_n)$. □

Recall that an algebra is semisimple if, and only if, it is representable as a subdirect product of simple algebras.

**Corollary 36.** *For each $n \geq 2$, the class of semisimple $n$-contractive* MTL*-algebras is the variety $\mathbb{S}_n\mathbb{MTL}$.*

For MV-algebras those varieties are easy to describe. By $Ł_n$ we will denote the usual finite MV-chain containing $n$ elements.

**Lemma 37.** *Let $\mathcal{A}$ be an* MV*-chain. The following are equivalent:*

*(i)* $\mathcal{A} \models (E_n)$.

*(ii)* $\mathcal{A} \in \mathbf{I}(\{Ł_1, \ldots, Ł_n\})$.

*(iii)* $\mathcal{A} \models (EM_n)$.

**Corollary 38.** *For each $n \geq 2$, $\mathbb{S}_n\mathbb{MTL} \cap \mathbb{MV} = \mathbb{C}_n\mathbb{MTL} \cap \mathbb{MV} = \mathbf{V}(\{Ł_1, \ldots, Ł_n\})$.*

However, in $\mathbb{MTL}$ and in $\mathbb{IMTL}$ the situation is not so easy. In the first level the varieties corresponding to $(E_n)$ and $(EM_n)$ are still easy to compute. Indeed, $\mathbb{S}_2\mathbb{MTL} = \mathbb{S}_2\mathbb{IMTL} = \mathbb{C}_2\mathbb{IMTL} = \mathbb{BA}$ and $\mathbb{C}_2\mathbb{MTL} = \mathbb{G}$. For $n = 3$, we have $\mathbb{S}_3\mathbb{MTL} \subsetneq \mathbb{WNM}$, in fact, the $\mathbb{S}_3$MTL-chains are those where the product of two non-one elements is always zero, i. e. the so-called drastic product. When $n = 3$, we also have $\mathbb{S}_3\mathbb{IMTL} = \mathbf{V}(Ł_3) = \mathbb{NM} \cap \mathbb{MV} \subsetneq \mathbb{NM} \subseteq \mathbb{C}_3\mathbb{IMTL}$. Therefore for each $n \geq 3$, $\mathbb{S}_n\mathbb{IMTL} \subsetneq \mathbb{C}_n\mathbb{IMTL}$ and

$\mathbb{S}_n\mathbb{MTL} \subsetneq \mathbb{C}_n\mathbb{MTL}$. The variety $\mathbb{S}_4\mathbb{IMTL}$ and its lattice of subvarieties have been studied in [17] .

Now we will show that the varieties of semisimple $n$-contractive chains are discriminator varieties.

**Definition 39.** *For every $n \geq 3$ we define a term $\delta_n(x, y) := (x \leftrightarrow y)^{n-1}$.*

Notice that if $a, b$ are elements in a simple $n$-contractive MTL-chain, then:

- $a = b$ if, and only if, $\delta_n(a, b) = \overline{1}^{\mathcal{A}}$.
- $a \neq b$ if, and only if, $\delta_n(a, b) = \overline{0}^{\mathcal{A}}$.

With this term we can define a discriminator just by considering $t(x, y, z) := (\delta(x, y) \wedge z) \vee (\neg\delta(x, y) \wedge x)$. It is clear that $t(x, y, z) = x$ if $x \neq y$, and $t(x, y, z) = z$ otherwise. In fact, Kowalski has proved that the only discriminator varieties of bounded integral commutative residuated lattice are those satisfying some of the $(EM_n)$ equations:

**Theorem 40** ([27])**.** *For every variety $\mathbb{K}$ of bounded integral commutative residuated lattices, the following are equivalent:*

*(i)* $\mathbb{K} \models (EM_n)$ *for some $n \geq 2$;*

*(ii)* $\mathbb{K}$ *is semisimple;*

*(iii)* $\mathbb{K}$ *is a discriminator variety.*

Using $\delta$ one can also give an equational definition of the class of algebras without negation fixpoint:

**Proposition 41.** *Let $\mathcal{A}$ be a simple $n$-contractive* MTL*-chain. Then, $\mathcal{A}$ has not the negation fixpoint if, and only if, $\mathcal{A} \models \delta_n(x, \neg x) \approx \overline{0}$.*

Finally, we will study the existence of atoms and coatoms in $n$-contractive chains, improving Proposition 32.

**Proposition 42.** *Let $\mathcal{A}$ be a non-trivial $n$-contractive* MTL*-chain. Then, $\mathcal{A}$ is subdirectly irreducible if, and only if, it has a coatom.*

*Proof:* One direction is true for every MTL-chain. Indeed, if $a \in A$ is a coatom, then the filter generated by $a$ is the minimum non-trivial filter. Conversely, let $F$ be the minimum non-trivial filter in $\mathcal{A}$. Then for all $a \in F \setminus \{\overline{1}^{\mathcal{A}}\}$ we have $a^{n-1} = \min F < \overline{1}^{\mathcal{A}}$. Suppose $\mathcal{A}$ has no coatom. Hence, $\bigvee_{a\in F} a = \overline{1}^{\mathcal{A}}$. But then $\min F = \bigvee_{a\in F} a^{n-1} = (\bigvee_{a\in F} a)^{n-1} = \overline{1}^{\mathcal{A}}$.
□

**Corollary 43.** *For every subdirectly irreducible $n$-contractive non-trivial* IMTL*-chain $\mathcal{A}$, there is $a \in A$ such that $a = \max A \setminus \{\overline{1}^{\mathcal{A}}\}$ and $\neg a = \min A \setminus \{\overline{0}^{\mathcal{A}}\}$.*

*Proof:* By Proposition 42 $\mathcal{A}$ has a coatom $a = \max A \setminus \{\overline{1}^{\mathcal{A}}\}$. Then $\neg a = \min A \setminus \{\overline{0}^{\mathcal{A}}\}$ because if $\overline{0}^{\mathcal{A}} < b < \neg a$ then $a < \neg b < \overline{1}^{\mathcal{A}}$. □

Proposition 42 also implies that the $n$-contractive MTL-chains defined by a left-continuous t-norm are not simple, i. e. there are no standard $\mathrm{S}_n$MTL-chains, which we already knew from the fact that there are even no subdirectly irreducible $n$-contractive standard chains.

## 4.2 Combining weakly cancellative and $n$-contractive fuzzy logics

In [29] the variety $\mathbb{WCMTL}$ of weakly cancellative MTL-algebras was defined to provide examples of indecomposable MTL-chains. Besides, the $\Omega$ operator gave rise to the variety $\Omega(\mathbb{WCMTL})$ which was a kind of analogue of $\mathbb{BL}$ in the sense that here all the chains were also decomposable as ordinal sums of weakly cancellative semihoops. Now it seems natural to consider the intersection of these varieties with the classes of $n$-contractive algebras (or equivalently the supremum of the corresponding logics) in order to obtain some new kinds of algebras with a nice and simpler structure. Therefore, we will consider for every $n \geq 2$ the logics $\mathrm{S}_n$WCMTL and $\mathrm{C}_n$WCMTL. Recall that WCMTL-chains satisfy the weak cancellation law saying that $x\&z = y\&z \neq \overline{0}$ implies $x = y$.

**Proposition 44.** *For every $n \geq 2$, $\{(WC), (C_n)\} \vdash_{\mathrm{MTL}} (S_n)$.*

*Proof:* Let $\mathcal{A}$ be an MTL-chain satisfying $(WC)$ and $(C_n)$. We will prove that it is simple. Take an arbitrary $a \in \mathcal{A} \setminus \{\overline{1}^{\mathcal{A}}\}$. Then, $a^{n-1}$ is an idempotent element, hence $a^{n-1}\&a^{n-1} = a^{n-1}\&\overline{1}^{\mathcal{A}} = a^{n-1}$. Since the chain is weakly cancellative, this implies $a^{n-1} = \overline{0}^{\mathcal{A}}$. Therefore, $\mathcal{A}$ is simple, i.e. satisfies $(S_n)$. □

**Corollary 45.** *Given any axiomatic extension* L *of* WCMTL *and $n \geq 2$, the extensions obtained by $(S_n)$ and $(C_n)$ coincide. In particular,* $\mathrm{S}_n\mathrm{WCMTL} = \mathrm{C}_n\mathrm{WCMTL}$.

It is straightforward to prove that the $\Omega$ operator and the schemata $(C_n)$ commute:

**Proposition 46.** *Let* L *be an axiomatic extension of* MTL*. For every $n \geq 2$,* $\Omega(\mathrm{C}_n\mathrm{L}) = \mathrm{C}_n\Omega(\mathrm{L})$.

Therefore, we have $\mathrm{C}_n\Omega(\mathrm{WCMTL}) = \Omega(\mathrm{C}_n\mathrm{WCMTL}) = \Omega(\mathrm{S}_n\mathrm{WCMTL})$.

Finally, we will consider for every $n \geq 2$ the logic $\Omega(\mathrm{S}_n\mathrm{MTL})$ and we will show that it is also finitely axiomatizable.

**Proposition 47.** *Let $\mathcal{A}$ be an $n$-contractive* MTL*-chain. The following are equivalent:*

*(i)* *$\mathcal{A}$ is totally decomposable.*

*(ii)* *$\mathcal{A}$ is an ordinal sum of simple $n$-contractive* MTL*-chains.*

*(iii)* $\mathcal{A} \models (y^{n-1} \to x) \vee (x \to x\&y) \approx \overline{1}$.

*Proof:* $(i) \Rightarrow (ii)$: If $\mathcal{A}$ is decomposable as the ordinal sum of its Archimedean classes, then, by Proposition 30, it is decomposable as ordinal sum of simple chains.
$(ii) \Rightarrow (iii)$: Take arbitrary elements $a, b \in A$. If $b \leq a$, then $b^{n-1} \leq a$, so they satisfy the equation. Suppose $a < b$. If they are in different components of the ordinal sum, then $a \to a\&b = \overline{1}^{\mathcal{A}}$. If they are in the same component, then, by simplicity, $b^{n-1} \leq a$.
$(iii) \Rightarrow (i)$: Suppose that $\mathcal{A}$ satisfies the equation. Take $a, b \in A \setminus \{\overline{1}^{\mathcal{A}}\}$ such that $a < b$ and they belong to different Archimedean classes. Then $b^{n-1} \to a \neq \overline{1}^{\mathcal{A}}$, so $a \to a\&b = \overline{1}^{\mathcal{A}}$, i. e. $a\&b = a$. Therefore, $\mathcal{A}$ is the ordinal sum of its Archimedean classes by Proposition 13. □

**Corollary 48.** *$\Omega(\mathbb{S}_n\mathbb{MTL})$ is the variety generated by the totally decomposable $n$-contractive chains, and it is axiomatized by $(y^{n-1} \to x) \vee (x \to x\&y) \approx \overline{1}$ and $x^n \approx x^{n-1}$.*

## 4.3 Methods for constructing MTL-chains

In this section we discuss three construction methods with MTL-chains and totaly ordered semihoops. These methods will be useful later. First, we recall an already known method (see [3]) and then we introduce two new ones.

Let $\mathcal{A} = \langle A, \&, \to, \wedge, \overline{1}\rangle$ be a totally ordered semihoop and $a \in A$. Then the truncation of $\mathcal{A}$ w.r.t. $a$ is an MTL-chain $\mathcal{A}_a = \langle [a, \overline{1}], \&_a, \to_a, \wedge, \vee, a, \overline{1}\rangle$, where $x \&_a y = (x\&y) \vee a$ and $\to_a$ is the restriction of $\to$ on the interval $[a, \overline{1}]$.

### First new method

Let $\mathcal{A} = \langle A, \&, \to, \wedge, \overline{1}\rangle$ be a totally ordered semihoop and $a \in A$. We define a new totally ordered semihoop $\mathcal{A}^a = \langle A \setminus (a, \overline{1}), \&^a, \to^a, \wedge, \overline{1}\rangle$ where $\&^a$ is just the restriction of & on $A \setminus (a, \overline{1})$, and $\rightarrow^a$ is defined as follows:

$$x \rightarrow^a y = \begin{cases} x \rightarrow y & \text{if } x \rightarrow y \in A \setminus (a, \overline{1})\,, \\ a & \text{if } x \rightarrow y \in (a, \overline{1})\,. \end{cases}$$

Clearly, $\mathcal{A}^a$ is a submonoid of $\mathcal{A}$. It is also not difficult to see that $\rightarrow^a$ is the residuum corresponding to $\&^a$.

Notice that if the monoidal reduct of the original algebra is cancellative, then the monoidal reduct of the resulting algebra is cancellative as well.

### Second new method

Let $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1}\rangle$ be an IMTL-chain and $a \in A$ such that $a \geq \neg a$. We define a new IMTL-chain $\mathcal{A}^a_{\neg a} = \langle \{\overline{0}\} \cup [\neg a, a] \cup \{\overline{1}\}, \&^a_{\neg a}, \rightarrow^a_{\neg a}, \wedge, \vee, \overline{0}, \overline{1}\rangle$ as follows:

$$x \&^a_{\neg a} y = \begin{cases} x \& y & \text{if } x \& y \in \{\overline{0}\} \cup [\neg a, a] \cup \{\overline{1}\}\,, \\ \neg a & \text{if } x \& y \in (\overline{0}, \neg a)\,. \end{cases}$$

$$x \rightarrow^a_{\neg a} y = \begin{cases} x \rightarrow y & \text{if } x \rightarrow y \in \{\overline{0}\} \cup [\neg a, a] \cup \{\overline{1}\}\,, \\ a & \text{if } x \rightarrow y \in (a, \overline{1})\,. \end{cases}$$

**Lemma 49.** *The algebra $\mathcal{A}^a_{\neg a}$ is an* IMTL*-chain.*

*Proof:* Let $B$ be the domain of $\mathcal{A}^a_{\neg a}$. We have to show that $\&^a_{\neg a}$ is commutative, associative, isotone, and $\overline{1}$ is the neutral element. The fact that $\overline{1}$ is neutral element and the commutativity are obvious. We will show that $\&^a_{\neg a}$ is isotone, i.e. $x \leq y$ implies $x \&^a_{\neg a} z \leq y \&^a_{\neg a} z$. There are several cases:

1. $x\&z, y\&z \in B$: then $x \&^a_{\neg a} z = x \& z \leq y \& z = y \&^a_{\neg a} z$.
2. $x\&z, y\&z \in (\overline{0}, \neg a)$: then $x \&^a_{\neg a} z = \neg a = y \&^a_{\neg a} z$.
3. $x\&z \in (\overline{0}, \neg a)$ and $y\&z \in B$: then $x \&^a_{\neg a} z = \neg a \leq y\&z = y \&^a_{\neg a} z$.
4. $x\&z \in B$ and $y\&z \in (\overline{0}, \neg a)$: then $x \&^a_{\neg a} z = x\&z = \overline{0} \leq y \&^a_{\neg a} z$.

Now we will prove that $\&^a_{\neg a}$ is associative. We can assume that $x, y, z < \overline{1}$ (i.e. $x, y, z \leq a$) otherwise it is obvious. We will check several cases:

1. $x\&y, y\&z, x\&y\&z \in B$: then $(x \&^a_{\neg a} y) \&^a_{\neg a} z = (x\&y) \&^a_{\neg a} z = x\&y\&z = x \&^a_{\neg a} (y\&z) = x \&^a_{\neg a} (y \&^a_{\neg a} z)$.
2. $x\&y, y\&z \in B$ and $x\&y\&z \in (\overline{0}, \neg a)$: then $(x \&^a_{\neg a} y) \&^a_{\neg a} z = (x\&y) \&^a_{\neg a} z = \neg a = x \&^a_{\neg a} (y\&z) = x \&^a_{\neg a} (y \&^a_{\neg a} z)$.

3. $x\&y \in (\overline{0}, \neg a)$ and $y\&z, x\&y\&z \in B$: then $x\&y\&z = \overline{0}$ since $x\&y\&z \leq x\&y$. Observe that $z \leq a$ since we assume $z < \overline{1}$. Thus $z\&\neg a = \overline{0}$. Consequently,

   $$(x\&^a_{\neg a}y)\&^a_{\neg a}z = \neg a\&^a_{\neg a}z = \overline{0} = x\&y\&z = x\&^a_{\neg a}(y\&z) = x\&^a_{\neg a}(y\&^a_{\neg a}z)\,.$$

4. The case $y\&z \in (\overline{0}, \neg a)$ and $x\&z, x\&y\&z \in B$ can be proved by commutativity and the previous case:

   $$(x\&^a_{\neg a}y)\&^a_{\neg a}z = z\&^a_{\neg a}(y\&^a_{\neg a}x) = (z\&^a_{\neg a}y)\&^a_{\neg a}x = x\&^a_{\neg a}(y\&^a_{\neg a}z)\,.$$

5. $x\&y, y\&z \in (\overline{0}, \neg a)$: then $(x\&^a_{\neg a}y)\&^a_{\neg a}z = \neg a\&^a_{\neg a}z = \overline{0} = x\&^a_{\neg a}\neg a = x\&^a_{\neg a}(y\&^a_{\neg a}z)$ since $x, z \leq a$.

6. $y\&z \in B$ and $x\&y, x\&y\&z \in (\overline{0}, \neg a)$: this case is not possible since we assume that $z < \overline{1}$. Indeed, $x\&y \in (\overline{0}, \neg a)$ implies $x\&y < \neg a$. Thus $x\&y\&z = \overline{0} \in B$ which is a contradiction.

7. The case $x\&y \in B$ and $y\&z, x\&y\&z \in (\overline{0}, \neg a)$ is also not possible as the previous one.

Now we have to show that $\&^a_{\neg a}$ is a residuated map. Observe that $x \to y \in (\overline{0}, \neg a)$ is not possible provided $x, y \in B$. Indeed, $x \to y \in (\overline{0}, \neg a)$ implies $y = \overline{0}$. Thus $\neg x < \neg a$. Since $\neg$ is involutive, we get $a < x = \overline{1}$. Hence $x \to y = \neg\overline{1} = \overline{0} \in B$ (a contradiction).

Finally we have to prove that $x\&^a_{\neg a}y \leq z$ iff $x \leq y \to^a_{\neg a} z$.

1. If $y \leq z$ then $y \to^a_{\neg a} z = y \to z = \overline{1}$ and $x\&^a_{\neg a}y \leq y \leq z$. Thus $x\&^a_{\neg a}y \leq z$ and $x \leq \overline{1} = y \to^a_{\neg a} z$ hold simultaneously.

2. Let $y > z$ and $x\&y = \overline{0}$. Then $y > \overline{0}$ and $\neg y < \overline{1}$. Further, observe that $y \to^a_{\neg a} z = a \wedge (y \to z)$ for $y > z$. Suppose that $x\&^a_{\neg a}y \leq z$. Then $x\&^a_{\neg a}y = x\&y = \overline{0}$ is equivalent to $x \leq \neg y$. Since $\neg y < \overline{1}$, we get $x \leq a$. Moreover, $\neg y = y \to \overline{0} \leq y \to z$. Thus $x \leq \neg y$ implies $x \leq a \wedge (y \to z) = y \to^a_{\neg a} z$. Now suppose that $x \leq y \to^a_{\neg a} z = a \wedge (y \to z)$. Since $x\&^a_{\neg a}y = x\&y = \overline{0} \leq z$ for any $z$, we are done.

3. Finally, let $y > z$ and $x\&y > \overline{0}$. Observe that in this case we have $y \to^a_{\neg a} z = a \wedge (y \to z)$ and $x\&^a_{\neg a}y = \neg a \vee (x\&y)$. Thus it is sufficient to prove that $\neg a \vee (x\&y) \leq z$ iff $x \leq a \wedge (y \to z)$. Suppose that $\neg a \vee (x\&y) \leq z$. Then $x\&y \leq z$ implies $x \leq y \to z$. Since $y \to z < \overline{1}$, we get $x \leq a$. Thus $x \leq a \wedge (y \to z)$. Conversely, assume that $x \leq a \wedge (y \to z)$. Then $x \leq y \to z$ implies $x\&y \leq z$. Since $x\&y > \overline{0}$, we get $\neg a \leq z$. Thus $\neg a \vee (x\&y) \leq z$.

□

## 4.4 Some properties of $n$-contractive fuzzy logics

In this section we will study some logical and algebraic properties of the considered logics.

### Local finiteness

First we focus our attention on local finiteness. The $n$-contractivity is a necessary condition for local finiteness:

**Proposition 50** ([29])**.** *Let $\mathbb{K} \subseteq \mathbb{MTL}$ be a variety. If $\mathbb{K}$ is locally finite, then there exists some $n \geq 2$ such that $\mathbb{K} \models x^n \approx x^{n-1}$.*

However, we will see that this condition is only necessary. Therefore for each variety of $n$-contractive MTL-algebras we have to discuss whether it is locally finite or not. It is straightforward that to show that $\mathbb{G}$ is locally finite. The property has been proved true also for $\mathbb{NM}$ in [16], and it has been generalized to $\mathbb{WNM}$ in [33].

**Theorem 51.** *$\Omega(\mathbb{S}_3\mathbb{MTL})$ is locally finite.*

*Proof:* It can be proved analogously to the case of $\mathbb{WNM}$. □

Finally, $\mathbb{S}_4\mathbb{IMTL}$ is also proved to be locally finite in [17]. Now we will present counterexamples showing that all the remaining varieties considered here fail to be locally finite.

Let $\mathbb{Z}$ denote the set of integers. Let $\mathbb{Z}^2_{\text{lex}}$ be the lexicographic product of two copies of the additive group of integers. Consider its negative cone $A = \{\langle x, y\rangle \in \mathbb{Z}^2_{\text{lex}} \mid \langle x, y\rangle \leq \langle 0, 0\rangle\}$, and take the algebra $\mathcal{A} = \langle A, +, \rightarrow, \wedge, \vee, \langle 0, 0\rangle\rangle$, where $\langle x_1, y_1\rangle \rightarrow \langle x_2, y_2\rangle = \langle x_2 - x_1, y_2 - y_1\rangle \wedge \langle 0, 0\rangle$. The algebra $\mathcal{A}$ forms a cancellative totally ordered semihoop. Then $\mathcal{A}^{\langle -1,0\rangle}$ is also cancellative totally ordered semihoop.

Now consider $(\mathcal{A}^{\langle -1,0\rangle})_{\langle -3,0\rangle}$. Then it is an $\text{S}_4$MTL-chain since $\langle -3, 0\rangle = \langle -1, 0\rangle^3$.

**Proposition 52.** *The subalgebra $\mathcal{C}$ of $(\mathcal{A}^{\langle -1,0\rangle})_{\langle -3,0\rangle}$ generated by $\{\langle -1, 0\rangle, \langle -1, -1\rangle\}$ is infinite.*

*Proof:* We will prove by induction that $\langle -1, -n\rangle \in C$ for each $n \in \mathbb{N}$. For $n = 0$ it is obvious since $\langle -1, 0\rangle$ is one of the generators. Assume that $\langle -1, -n\rangle \in C$. Then

$$\langle -1, -n - 1\rangle = \langle -1, 0\rangle \rightarrow (\langle -1, -1\rangle \& \langle -1, -n\rangle)\,.$$

□

**Theorem 53.** *The varieties $\mathbb{C}_n\mathbb{MTL}$, $\mathbb{S}_n\mathbb{MTL}$, $\Omega(\mathbb{S}_n\mathbb{MTL})$, $\mathbb{S}_n\mathbb{WCMTL}$ and $\Omega(\mathbb{S}_n\mathbb{WCMTL})$ for each $n \geq 4$ are not locally finite.*

*Proof:* The fact for $\mathbb{C}_n\mathbb{MTL}$ and $\mathbb{S}_n\mathbb{MTL}$ follows from the previous proposition. Then, since the statement holds for $\mathbb{S}_n\mathbb{MTL}$, it must hold also for the variety $\Omega(\mathbb{S}_n\mathbb{MTL})$. As the counterexample arose from the truncation of a cancellative totally ordered semihoop, it must satisfy the weak cancellation property. Thus it belongs, in fact, to $\mathbb{S}_4\mathbb{WCMTL}$. Hence $\mathbb{S}_n\mathbb{WCMTL}$ and $\Omega(\mathbb{S}_n\mathbb{WCMTL})$ are not locally finite as well. □

We give now a counterexample for the involutive cases. Take $\mathcal{A}$ as before and consider its truncation $\mathcal{B} = \mathcal{A}_{\langle -4,0\rangle}$, which is an IMTL-chain (in fact, it is an MV-chain). We will apply the second construction to this algebra obtaining $\mathcal{B}^{\langle -1,0\rangle}_{\neg\langle -1,0\rangle}$. Then the algebra $\mathcal{B}^{\langle -1,0\rangle}_{\neg\langle -1,0\rangle}$ is an IMTL-chain and moreover, it is an $\mathrm{S}_5$IMTL-chain since $\langle -1,0\rangle^4 = \langle -4,0\rangle$. Besides, the subalgebra generated by $\{\langle -1,0\rangle, \langle -1,-1\rangle\}$ is infinite. The proof is same as of Proposition 52.

**Theorem 54.** *The varieties $\mathbb{C}_n\mathbb{IMTL}$ and $\mathbb{S}_n\mathbb{IMTL}$ are not locally finite for each $n \geq 5$.*

Finally, we will deal with the remaining cases $\mathbb{C}_3\mathbb{MTL}$, $\mathbb{C}_3\mathbb{IMTL}$ and $\mathbb{C}_4\mathbb{IMTL}$. A totally ordered commutative semigroup $\mathcal{S} = \langle S, \&, \leq\rangle$ is a commutative semigroup satisfying:

1. $\langle S, \leq\rangle$ is a chain,
2. $x \leq y$ implies $x\&z \leq y\&z$ for all $x, y, z \in S$.

Moreover, if $x\&y \leq y$ holds for all $x, y \in S$ then $\mathcal{S}$ is said to be negative.

Let $\mathbb{Z}^-_\infty$ be the set of non-positive integers together with $-\infty$ and $A_i$, $i = 1, 2, 3$ be the three disjoint copies of $\mathbb{Z}^-_\infty$. We will denote an integer $x$ from $A_i$ by $x_i$. Now let $S = A_1 \cup A_2 \cup A_3$. We introduce an order on $S$ by $x_i \leq y_j$ if either $i > j$ or ($i = j$ and $x \leq y$). Let us define a binary operation & on $S$ as follows:

$$x_i\&y_j = \begin{cases} -\infty_1 & \text{if } i = j = 1\,, \\ (x+y)_3 & \text{if } i + j = 3\,, \\ -\infty_3 & \text{otherwise.} \end{cases}$$

**Lemma 55.** *The algebra $\mathcal{S} = \langle S, \&, \leq\rangle$ is a totally ordered negative commutative semigroup.*

*Proof:* The operation & is clearly commutative. We have to check that it is associative, i.e., $(x_i\&y_j)\&z_k = x_i\&(y_j\&z_k)$. There are several cases:

1. If $i = j = k = 1$ then both sides equal $-\infty_1$.

2. If $\min\{i,j,k\} = 3$ then both sides equal $-\infty_3$.

3. If at least two of $\{i,j,k\}$ equal 2 then both sides equal $-\infty_3$.

4. If $i = j = 1$ and $k = 2$ then we get

$$(x_1\&y_1)\&z_2 = -\infty_1\&z_2 = (-\infty+z)_3 = -\infty_3 = x_1\&(y+z)_3 = x_1\&(y_1\&z_2)\,.$$

5. The case $i = 2$ and $j = k = 1$ can be obtained by commutativity and the previous case.

6. If $i = k = 1$ and $j = 2$ then we get

$$(x_1\&y_2)\&z_1 = (x+y)_3\&z_1 = -\infty_3 = x_1\&(y+z)_3 = x_1\&(y_2\&z_1)\,.$$

Finally we have to check that & is isotone. Let $x_i \leq y_j$ and $z_k \in S$. If $i+k > 3$ then $x_i\&z_k = -\infty_3 \leq y_j\&z_k$. Suppose that $i+k = 2$. Then $x_i\&z_k = -\infty_1 = y_j\&z_k$ since $i \geq j$. Finally assume that $i+k = 3$. Then $x_i\&z_k = (x+z)_3$. If $j+k < 3$ then $y_j\&z_k = -\infty_1 \geq (x+z)_3$. If $j+k = 3$ then $i = j$ and $x \leq y$. Thus $(x+z)_3 \leq (y+z)_3$. The fact that $S$ is negative can be easily checked. □

It is known that each totally ordered negative commutative semigroup without a neutral element can be extended to a totally ordered commutative monoid. Let $M = S \cup \{e\}$ and define $x\&e = x$, $x \leq e$ for all $x \in M$. Thus $\mathcal{M} = \langle M, \&, \leq, e\rangle$ becomes an integral totally ordered commutative monoid. Moreover, $M$ is clearly inversely well-ordered. Thus each nonempty subset of $M$ has a maximum and we can introduce a residuum $\rightarrow$. Consequently, the algebra $\mathcal{M} = \langle M, \&, \rightarrow, \leq, -\infty_3, e\rangle$ is an MTL-chain. It is not difficult to see that $\mathcal{M}$ is even a $\mathrm{C_3MTL}$-chain.

**Lemma 56.** *The subalgebra $\mathcal{A}$ of $\mathcal{M}$ generated by $\{0_1, 0_2, -1_3\}$ is infinite.*

*Proof:* We will prove that there is an infinite decreasing sequence in $M$. Let $x_3$ is in $A$. There is at least one such element since $-1_3$ is one of the generators. We will show that there is $y_3 < x_3$. Take $a = 0_1 \rightarrow x_3 = x_2$ and $b = 0_2 \rightarrow x_3 = x_1$. Then $y_3 = a\&b = x_2\&x_1 = (2x)_3 < x_3$. □

**Theorem 57.** *The varieties $\mathbb{C}_3\mathbb{MTL}$, $\mathbb{C}_3\mathbb{IMTL}$ and $\mathbb{C}_4\mathbb{IMTL}$ are not locally finite.*

*Proof:* For the first part take the above-mentioned $\mathrm{C_3MTL}$-chain $\mathcal{M}$ whose finitely generated subalgebra is infinite. The other statements can be proved by taking the disconnected rotation[5] of $\mathcal{M}$ which is a $\mathrm{C_3IMTL}$-chain. Since its subalgebra generated by $\{0_1, 0_2, -1_3\}$ is infinite as well, we are done. □

[5] The disconnected rotation is a construction method how to produce from a totally ordered semihoop an IMTL-chain, for details see [30].

### 4.4.1 Finite embeddability property, finite model property and decidability

We turn now to the finite embeddability property. It was proved for the variety of bounded commutative integral residuated lattices by Blok and Van Alten in [5], and by using the same construction Ono[6] proved the FEP also for $\mathbb{MTL}$, $\mathbb{IMTL}$, $\mathbb{SMTL}$, $\mathbb{C}_n\mathbb{MTL}$ and $\mathbb{C}_n\mathbb{IMTL}$, for every $n \geq 2$. The proof has been improved in [10].

**Proposition 58.** $\mathbb{MTL} = \bigvee_{n\geq 2} \mathbb{C}_n\mathbb{MTL}$ *and* $\mathbb{IMTL} = \bigvee_{n\geq 2} \mathbb{C}_n\mathbb{IMTL}$.

*Proof:* $\mathbb{MTL}$ and $\mathbb{IMTL}$ have the FEP, therefore they are generated by their finite chains. Since all finite MTL-chains are $n$-contractive for some $n$, we obtain that $\mathbb{MTL}$ is generated by the $n$-contractive MTL-chains and $\mathbb{IMTL}$ is generated by the $n$-contractive IMTL-chains. □

**Theorem 59.** *For every $n \geq 2$, the following varieties have the FEP:*

- $\mathbb{S}_n\mathbb{MTL}$
- $\mathbb{S}_n\mathbb{IMTL}$
- $\Omega(\mathbb{S}_n\mathbb{MTL})$
- $\mathbb{S}_n\mathbb{WCMTL}$
- $\Omega(\mathbb{S}_n\mathbb{WCMTL})$

*Proof:* Let $\mathbb{L}$ be any variety from the class $\{\mathbb{S}_n\mathbb{MTL}, \Omega(\mathbb{S}_n\mathbb{MTL}), \mathbb{S}_n\mathbb{WCMTL}, \Omega(\mathbb{S}_n\mathbb{WCMTL}) \mid n \geq 2\}$. Let $\mathcal{A}$ be an arbitrary L-chain and $B \subseteq A$ be a finite subset of its carrier. Consider the monoid $\mathcal{M}$ generated by $B \cup \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$, i. e. the submonoid of $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ obtained by closing $B \cup \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$ under $\&^{\mathcal{A}}$. By the simplicity of $\mathcal{A}$, $M$ is finite, so it is residuated. Expanding $\mathcal{M}$ with the residuum it becomes an MTL-chain. Observe that, in fact, $\mathcal{M}$ is an L-chain, since the required properties are preserved when we generate the monoid. Finally, it can be proved that the identity mapping $id : B \to M$ is the embbeding of $\mathcal{B}$ into $\mathcal{M}$ (the proof is same as the proof of [22, Lemma 3.3]). Thus $\mathbb{L}$ has the FEP.

As regards to $\mathbb{S}_n\mathbb{IMTL}$, we follow the method given in [10] to prove the FEP of IMTL. Let $\mathcal{A}$ be an $\mathrm{S}_n$IMTL-chain and $\mathcal{B}$ be a finite partial subalgebra of $\mathcal{A}$. Without any loss of generality we can assume that $\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \in B$ and $B$ is closed under $\neg^{\mathcal{A}}$. Consider the submonoid $\mathcal{M}$ of $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ generated by $B$. Then again the monoid $\mathcal{M}$ can be enriched by a residuum so it forms an

[6] Private communication.

MTL-chain $\langle M, \&^{\mathcal{M}}, \rightarrow^{\mathcal{M}}, \wedge^{\mathcal{M}}, \vee^{\mathcal{M}}, \overline{0}^{\mathcal{M}}, \overline{1}^{\mathcal{M}}\rangle$ in which $\mathcal{B}$ can be embedded by the identity mapping. Now let us define the following subset of $M$:

$$M \rightarrow^{\mathcal{M}} B = \{m \rightarrow^{\mathcal{M}} b : m \in M,\ b \in B\}\,.$$

In [10] it is proved that $M \rightarrow^{\mathcal{M}} B$ forms a finite IMTL-chain in which $\mathcal{B}$ can be embedded. Moreover, it is proved that $M \rightarrow^{\mathcal{M}} B$ is the &-free subreduct of $\mathcal{M}$. Since the $(\mathrm{S}_n)$ schema can be equivalently rewritten in terms of $\rightarrow$, $\vee$ and $\overline{0}$: $\varphi \vee (\varphi \rightarrow (\varphi \rightarrow \cdots (\varphi \rightarrow \overline{0}) \cdots))$, this schema is still satisfied in $M \rightarrow^{\mathcal{M}} B$. Thus it must be an $\mathrm{S}_n$IMTL-chain. □

### 4.4.2 Standard completeness properties

Finally, we consider the standard completeness properties for $n$-contractive fuzzy logics. As mentioned above, the SSC was proved by Ciabattoni, Esteva and Godo in [9] for $\mathrm{C}_n$MTL and $\mathrm{C}_n$IMTL for every $n \geq 2$.

**Theorem 60.** *For every $n \geq 2$, we have:*

*(a)* $\mathrm{S}_n$MTL, $\mathrm{S}_n$IMTL *and* $\mathrm{S}_n$WCMTL *do not enjoy SC because there are no standard algebras in the corresponding variety.*

*(b)* $\Omega(\mathrm{S}_n\mathrm{MTL})$ *enjoys the SSC.*

*Proof:* $(a)$: It follows from Proposition 42.
$(b)$: We will prove it by using the embedding method of Jenei and Montagna (see [23]). Let $\mathcal{A}$ be a countable $\Omega(\mathrm{S}_n\mathrm{MTL})$-chain. Consider the following set:

$X := \{\langle s, q\rangle : s \in A, s \neq \overline{0}^{\mathcal{A}}, q \in \mathbb{Q} \cap (0,1]\} \cup \{\langle \overline{0}^{\mathcal{A}}, 1\rangle\}$, endowed with the lexicographical order and the operation

$$\langle s, q\rangle \circ \langle s', q'\rangle := \begin{cases} \min\{\langle s, q\rangle, \langle s', q'\rangle\} & \text{if } s\&^{\mathcal{A}}s' = \min\{s, s'\} \\ \langle s\&^{\mathcal{A}}s', 1\rangle & \text{otherwise.} \end{cases}$$

It was shown in [23] that $X$ together with $\circ$ forms a countable MTL-chain which is order-isomorphic to $\mathbb{Q} \cap [0,1]$. Thus it is sufficient to prove that it is an $\Omega(\mathrm{S}_n\mathrm{MTL})$-chain as well. By Corollary 48 we have to check that the identity $(y^{n-1} \rightarrow x) \vee (x \rightarrow x\&y) \approx \overline{1}$ is valid in $X$. Let $\langle s, q\rangle, \langle s', q'\rangle \in X$ be such that $\langle s, q\rangle < \langle s', q'\rangle^{n-1}$. We must prove that $\langle s, q\rangle \circ \langle s', q'\rangle = \langle s, q\rangle$.

We have:

$$\langle s', q'\rangle^{n-1} = \begin{cases} \langle s', q'\rangle & \text{if } (s')^2 = s' \\ \langle (s')^{n-1}, 1\rangle & \text{otherwise.} \end{cases}$$

In both cases the first component of $\langle s', q' \rangle^{n-1}$ equals $(s')^{n-1}$. Therefore, $s \leq (s')^{n-1}$. Since $\mathcal{A}$ is totally decomposable $n$-contractive MTL-chain, we have $s\&s' = s = \min\{s, s'\}$. Thus $\langle s, q \rangle \circ \langle s', q' \rangle = \langle s, q \rangle$. It can be also easily seen that $\circ$ satisfies the $n$-contraction law.

The final step in the proof of standard completeness theorem from [23] is the extension of the operation $\circ$ in $X$ (viewed as an algebra over $\mathbb{Q} \cap [0,1]$) to the whole interval $[0,1]$. Consider the completion of $\circ$ in $[0,1]$: for every $a, b \in [0,1]$, define $a \otimes b := \sup\{q \circ p : q \leq a, p \leq b,\ q, p \in \mathbb{Q}\}$. It was shown in [23] that $[0,1]$ together with $\otimes$ forms a standard MTL-chain. Thus it is again sufficient to prove that it is also $\Omega(\mathrm{S}_n\mathrm{MTL})$-chain. By Corollary 48 we have to check the validity of the identity $(y^{n-1} \to x) \vee (x \to x\&y) \approx \overline{1}$ (the validity of the $n$-contractivity needs not to be checked since in [9] it has been already proved that this construction preserves it). Let $a, b \in [0,1]$ be such that $a < b^{n-1}$. We must prove that $a \otimes b = a$. Obviously, it is sufficient to prove $a \leq a \otimes b$. Since $\otimes$ is left-continuous, we have

$$b^{n-1} = (\sup\{p \in \mathbb{Q} : p \leq b\})^{n-1} = \sup\{p^{n-1} \in \mathbb{Q} : p \leq b\}\,.$$

Thus there exists an element $z \in \mathbb{Q}$ such that $z \leq b$ and $a < z^{n-1}$. Then we have

$$\begin{aligned} a \otimes b &= \sup\{q \circ p : q \leq a,\ p \leq b,\ q, p \in \mathbb{Q}\} = \\ &= \sup\{q \circ p : q \leq a,\ z \leq p \leq b,\ q, p \in \mathbb{Q}\} \geq \\ &\geq \sup\{q \circ z : q \leq a,\ q \in \mathbb{Q}\} = \\ &= \sup\{q : q \leq a,\ q \in \mathbb{Q}\} = a\,. \end{aligned}$$

□

Finally, we will prove that the logic of ordinal sums of $n$-contractive WCMTL-chains enjoys the FSSC but not the SSC.

**Theorem 61.** $\Omega(\mathrm{S}_n\mathrm{WCMTL})$ *enjoys the FSSC.*

*Proof:* The first part of the proof follows the proof of the standard completeness theorem published in [21, Section 4]. Take a finite set $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $T \nvdash_{\Omega(\mathrm{S}_n\mathrm{WCMTL})} \varphi$. Then, there is an $\Omega(\mathrm{S}_n\mathrm{WCMTL})$-chain $\mathcal{A}$ and an evaluation $e : Fm_{\mathcal{L}} \to \mathcal{A}$ such that $e[T] \subseteq \{\overline{1}^{\mathcal{A}}\}$ and $e(\varphi) \neq \overline{1}^{\mathcal{A}}$. Consider the set $G := \{e(\psi) : \psi$ is a subformula of some formula of $T \cup \{\varphi\}\}$. $G$ is finite because $T$ is. Let $\mathcal{A}'$ be the $\to$-free reduct of $\mathcal{A}$ and $\mathcal{S}$ be the subalgebra of $\mathcal{A}'$ generated by $G$. It was shown in [21] that $\mathcal{S} = \langle S, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{S}}, \overline{1}^{\mathcal{S}} \rangle$ forms a countable inversely well-ordered MTL-chain (with a finite number of Archimedean classes) and $T \not\models_{\mathcal{S}} \varphi$.

The fact that & is just the restriction of the monoidal operation of $\mathcal{A}$ has several consequences. First, since $\mathcal{A}$ is totally decomposable by Proposition 47, it follows by Proposition 13 that $\mathcal{S}$ is totally decomposable as well.

Secondly, $\mathcal{S}$ is also $n$-contractive as $\mathcal{A}$ is. Thus, by Proposition 47, $\mathcal{S}$ is an ordinal sum of simple $n$-contractive MTL-chains. Thirdly, each component in the ordinal sum also remains weakly cancellative. Thus $\mathcal{S}$ is an $\Omega(\mathrm{S}_n\mathrm{WCMTL})$-chain. Since $\mathcal{S}$ has finitely many Archimedean classes, this ordinal sum must have a finite number of components, say $\mathcal{S} = \bigoplus_{i<k} \mathcal{C}_i$ for some natural number $k$.

Now we extend $\mathcal{S}$, so that it becomes order-isomorphic to $[0, 1]$. We start with the particular components $\mathcal{C}_i$.

**Lemma 62.** *Each $\mathcal{C}_i$ can be extended into an* $\mathrm{S}_n\mathrm{WCMTL}$*-chain which is order-isomorphic to the subset of reals* $[0, a] \cup \{1\}$ *for any* $a \in (0, 1)$*.*

*Proof:* Since each $\mathcal{C}_i$ forms an $\mathrm{S}_n\mathrm{WCMTL}$-chain, each $\mathcal{C}_i$ has a coatom $c_i$ by Proposition 42.

Now we use the same method as in [29, Theorem 46] and define a new chain over the set

$$X := \{\langle s, r\rangle : s \in \mathcal{C}_i \setminus \{\overline{0}^{\mathcal{C}_i}\}, r \in (0, 1]\} \cup \{\langle \overline{0}^{\mathcal{C}_i}, 1\rangle\},$$

with the lexicographical order $\leq_{lex}$ and the following operations:

$$\langle a, x\rangle \circ \langle b, y\rangle = \begin{cases} \langle \overline{0}^{\mathcal{C}_i}, 1\rangle & \text{if } a \&^{\mathcal{C}_i} b = \overline{0}^{\mathcal{C}_i}, \\ \langle a \&^{\mathcal{C}_i} b, xy\rangle & \text{otherwise.} \end{cases}$$

$$\langle a, x\rangle \Rightarrow \langle b, y\rangle = \begin{cases} \langle a \to^{\mathcal{C}_i} b, 1\rangle & \text{if } a \&^{\mathcal{C}_i}(a \to^{\mathcal{C}_i} b) < b, \\ \langle a \to^{\mathcal{C}_i} b, \min\{1, y/x\}\rangle & \text{otherwise.} \end{cases}$$

Then $\mathcal{X} = \langle X, \circ, \Rightarrow, \leq_{lex}, \langle \overline{0}^{\mathcal{C}_i}, 1\rangle, \langle \overline{1}^{\mathcal{C}_i}, 1\rangle\rangle$ is a WCMTL-chain as it was shown in the proof of [29, Theorem 46].

Now we apply one of the construction methods introduced in Section 4.3 on $\mathcal{X}$ in order to obtain an $\mathrm{S}_n\mathrm{WCMTL}$-chain. We claim that $\mathcal{X}^{\langle c_i, 1\rangle}$ is an $\mathrm{S}_n\mathrm{WCMTL}$-chain. Clearly, it is a WCMTL-chain. Moreover, it is simple and $n$-contractive since $\langle c_i, 1\rangle^{n-1} = \langle c_i^{n-1}, 1\rangle = \langle \overline{0}^{\mathcal{C}_i}, 1\rangle$. Futher, it is easy to prove that the mapping $f : C_i \to X^{\langle c_i, 1\rangle}$ defined by $f(s) = \langle s, 1\rangle$ is an embedding of $\mathcal{C}_i$ into $\mathcal{X}^{\langle c_i, 1\rangle}$.

Finally, we have to show that $X^{\langle c_i, 1\rangle}$ is order-isomorphic to $[0, a] \cup \{1\}$ for any $a \in (0, 1)$. Clearly it is sufficient to prove that $X^{\langle c_i, 1\rangle} \setminus \{\langle \overline{1}^{\mathcal{C}_i}, 1\rangle\}$ is order-isomorphic to $[0, a]$. It is well-known from the set theory that a totally ordered set $W$ is order-isomorphic to $[0, a]$ if it is complete, contains a minimum and a maximum, and has a countable subset which is dense in it. Clearly, $\langle \overline{0}^{\mathcal{C}_i}, 1\rangle$ (resp. $\langle c_i, 1\rangle$) is the minimum (resp. maximum). The set $\{\langle s,r\rangle \in X^{\langle c_i,1\rangle} \setminus \{\langle \overline{1}^{\mathcal{C}_i},1\rangle\} : r \in \mathbb{Q}\}$ is obviously a countable dense subset in $X^{\langle c_i,1\rangle} \setminus \{\langle \overline{1}^{\mathcal{C}_i},1\rangle\}$ since $\mathcal{C}_i$ is countable. Recall that each $\mathcal{C}_i$ is inversely well-ordered because $\mathcal{S}$ is. Now, let $\emptyset \neq Z \subseteq X^{\langle c_i,1\rangle} \setminus \{\langle \overline{1}^{\mathcal{C}_i},1\rangle\}$. Then $\sup \pi_1(Z)$ exists since $\mathcal{C}_i$ is inversely well-ordered. Let us denote it by $z$. Then $\sup Z = \langle z,\alpha\rangle$ where $\alpha = \sup\{x \in (0,1] : \langle z,x\rangle \in Z\}$. Thus $X^{\langle c_i,1\rangle} \setminus \{\langle \overline{1}^{\mathcal{C}_i},1\rangle\}$ is complete. □

Now we finish the proof of Theorem 61. Let $\mathcal{G}_i = \langle (0,1], \min, \rightarrow, \min, 1\rangle$, for each $0 \leq i < k$ be the totally ordered semihoop in which the monoidal operation is the minimum operation. The extended version of each $\mathcal{C}_i$ from Lemma 62 will be denoted by $\mathcal{C}'_i$. Then we take the following ordinal sum:

$$\mathcal{B} = \mathcal{C}'_0 \oplus \mathcal{G}_0 \oplus \cdots \oplus \mathcal{C}'_{k-1} \oplus \mathcal{G}_{k-1}\,.$$

Then $\mathcal{B}$ is an $\Omega(\mathrm{S}_n\mathrm{WCMTL})$-chain since each $\mathcal{C}_i$ is an $\mathrm{S}_n\mathrm{WCMTL}$-chain and each $\mathcal{G}_i$ can be viewed as an infinite ordinal sum of two element Boolean algebras. It is straightforward to check that $\mathcal{S} = \bigoplus_{i<k} \mathcal{C}_i$ can be embedded in $\mathcal{B}$ and $\mathcal{B}$ is order-isomorphic to $[0,1]$. □

Although $\Omega(\mathrm{S}_n\mathrm{WCMTL})$ enjoys the FSSC, it does not enjoy the SSC as it is shown in the rest of this section. Let $p$, $q$, and $r_i$, $i \in \mathbb{N}$, be propositional variables. We define the following set of formulae:

$$\Gamma = \{p \rightarrow q, r_0 \leftrightarrow q^2\} \cup \{r_i \rightarrow p,\ r_{i+1} \leftrightarrow (p \rightarrow r_i \& q),\ r_{i+1}\&p \leftrightarrow r_i\&q \mid i \in \mathbb{N}\}\,.$$

**Lemma 63.** *For each standard* $\Omega(\mathrm{WCMTL})$*-chain* $\mathcal{A}$ *we have* $\Gamma \models_{\mathcal{A}} r_2 \rightarrow r_1$.

*Proof:* Let $e$ be an evaluation on $\mathcal{A}$ such that $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$. We set $x = e(p)$, $y = e(q)$, and $a_i = e(r_i)$, $i \in \mathbb{N}$. Then clearly $x \leq y$ since $p \rightarrow q \in \Gamma$. Thus we have $a_i\&x \leq a_i\&y$ which is equivalent to $a_i \leq x \rightarrow a_i\&y = a_{i+1}$. Hence the sequence $\{a_i\}_{i\in\mathbb{N}}$ is non-decreasing. The order of evaluations of all considered variables has to be the following:

$$y^2 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq \cdots \leq x \leq y\,.$$

The first equality holds since $r_0 \leftrightarrow q^2 \in \Gamma$. It is obvious that $x, y, a_i$ belong to the same Archimedean class. Hence also to the same component in the ordinal sum, say $\mathcal{A}_k$. The algebra $\mathcal{A}_k$ is a weakly cancellative totally ordered semihoop. If it has a bottom element then we denote it by $\overline{0}_k$.

As $\mathcal{A}$ is complete, all suprema exist. Let $a = \bigvee a_i$. From the fact that $r_{i+1}\&p \leftrightarrow r_i\&q \in \Gamma$ for each $i \in \mathbb{N}$, we get $a_{i+1}\&x = a_i\&y$ for each $i \in \mathbb{N}$. Consequently, $\bigvee(a_{i+1}\&x) = \bigvee(a_i\&y)$. Since & distributes over the joins, we obtain $a\&x = a\&y$. Now, suppose that $a\&y \neq \overline{0}_k$. Then we can use the weak cancellation law and get $x = y$. Thus $a_2\&y = a_2\&x = a_1\&y$ since $r_{i+1}\&p \leftrightarrow r_i\&q \in \Gamma$. If $a_2\&y \neq \overline{0}_k$ then $a_2 = a_1$ by the weak cancellation law, i.e. $e(r_2 \to r_1) = \overline{1}^{\mathcal{A}}$. If $a_2 \& y = \overline{0}_k$ then $a_1 \& y = \overline{0}_k$, and $a_0 \& y = \overline{0}_k$ since $a_i \& y \leq a_j \& y$ for $i \leq j$. For $r_{i+1} \leftrightarrow (p \to r_i \& q) \in \Gamma$, we get $a_1 = x \to a_0 \& y = x \to \overline{0}_k = x \to a_1 \& y = a_2$.

Similarly, if $a \& y = \overline{0}_k$ then all $a_i \& y = \overline{0}_k$ and we can use the same argument as before. □

**Lemma 64.** *There is an* $\mathrm{S_4WCMTL}$*-chain* $\mathcal{B}$ *such that* $\Gamma \not\models_{\mathcal{B}} r_2 \to r_1$*, i.e.* $\Gamma \not\vdash_{\mathrm{S_4WCMTL}} r_2 \to r_1$.

*Proof:* Consider the lexicographic product of two copies of the additive group of integers. Let $\mathcal{A}$ denote its negative cone. Then $\mathcal{A} = \langle A, \&, \to, \leq, \langle 0, 0 \rangle \rangle$ forms an integral commutative cancellative residuated chain. We have $\langle x_1, y_1 \rangle \& \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$. Now consider an algebra $\mathcal{B} = (\mathcal{A}^{\langle -1,0 \rangle})_{\langle -3,0 \rangle}$ arising from $\mathcal{A}$ by cutting off the open interval $(\langle -1, 0 \rangle, \langle 0, 0 \rangle)$ and then truncated at $\langle -3, 0 \rangle$. The resulting algebra $\mathcal{B}$ is an $\mathrm{S_4WCMTL}$-chain. Let $e$ be an evaluation such that $e(p) = \langle -1, -1 \rangle$, $e(q) = \langle -1, 0 \rangle$, and $e(r_i) = \langle -2, i \rangle$, $i \in \mathbb{N}$. Then clearly $e(p) \leq e(q)$, $e(r_0) = e(q^2)$, and $e(r_i) \leq e(p)$ for all $i \in \mathbb{N}$. Further, we have

$$e(p \to r_i \& q) = \langle -1, -1 \rangle \to \langle -2, i \rangle \& \langle -1, 0 \rangle = \\ = \langle -1, -1 \rangle \to \langle -3, i \rangle = \langle -2, i + 1 \rangle = e(r_{i+1}) \,.$$

Finally,

$$e(r_{i+1} \& p) = \langle -2, i + 1 \rangle \& \langle -1, -1 \rangle = \langle -3, i \rangle = \langle -2, i \rangle \& \langle -1, 0 \rangle = e(r_i \& q) \,.$$

Thus $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{B}}\}$ but $e(r_2) = \langle -2, 2 \rangle > \langle -2, 1 \rangle = e(r_1)$. Thus $\Gamma \not\models_{\mathcal{B}} r_2 \to r_1$. □

**Theorem 65.** *Any logic between* $\Omega(\mathrm{WCMTL})$ *and* $\mathrm{S_4WCMTL}$ *does not enjoy the SSC.*

# 5 Conclusions

A new hierarchy of fuzzy logics has been defined in this paper by using the axioms of $n$-contraction, the generalized excluded middle axioms, the weak cancellation axiom and the $\Omega$ operator. We have studied some of their logical and algebraic properties. The obtained results are gathered in the Table 4.


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| | LF | FEP | FMP | Decidable | SC | FSSC | SSC |
|---|---|---|---|---|---|---|---|
| G | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| WNM | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| NM | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| $C_n$MTL, $n \geq 3$ | No | Yes | Yes | Yes | Yes | Yes | Yes |
| $C_n$IMTL, $n \geq 3$ | No | Yes | Yes | Yes | Yes | Yes | Yes |
| $S_n$MTL, $n \geq 4$ | No | Yes | Yes | Yes | No | No | No |
| $S_4$IMTL | Yes | Yes | Yes | Yes | No | No | No |
| $S_n$IMTL, $n \geq 5$ | No | Yes | Yes | Yes | No | No | No |
| $\Omega(S_3$MTL) | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| $\Omega(S_n$MTL), $n \geq 4$ | No | Yes | Yes | Yes | Yes | Yes | Yes |
| $S_n$WCMTL, $n \geq 4$ | No | Yes | Yes | Yes | No | No | No |
| $\Omega(S_n$WCMTL), $n \geq 4$ | No | Yes | Yes | Yes | Yes | Yes | No |

Table 4: Algebraic and logical properties of $n$-contractive fuzzy logics.

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