On the Structure of Finite Integral Commutative Residuated Chains

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Abstract

Among the class of finite integral commutative residuated chains (ICRCs), we identify those algebras which can be obtained as a nuclear retraction of a conuclear contraction of a totally ordered Abelian ℓ-group. We call the ICRCs satisfying this condition regular. Then we discuss the structure of finite regular ICRCs. Finally, we prove that the class of regular members generate a strictly smaller variety than the variety generated by ICRCs.

1 Introduction

The variety $\mathcal{ICRL}^{\mathcal{C}}$ of representable integral commutative residuated lattices forms an equivalent algebraic semantics for the false-free fragment of the weakest t-norm based fuzzy logic MTL (see [5]). Due to this connection, it would be very useful to understand the structure of algebras from $\mathcal{ICRL}^{\mathcal{C}}$ if we want to study the logical properties of MTL, for instance its computational complexity. The structural classification of general members of $\mathcal{ICRL}^{\mathcal{C}}$ seems to be a complex task. Thus we focus in this paper mainly on the structure of totally ordered finite algebras from $\mathcal{ICRL}^{\mathcal{C}}$. These algebras are important since they generate the variety $\mathcal{ICRL}^{\mathcal{C}}$ (see [3, 4]).

There are several structural results on particular subclasses of $\mathcal{ICRL}^{\mathcal{C}}$ in the literature. Many of them are based on a connection with Abelian ℓ-groups. For instance, any cancellative member $\mathbf{A}$ of $\mathcal{ICRL}^{\mathcal{C}}$ can be viewed as a σ-contraction of an Abelian ℓ-group $\mathbf{G}$ for a conucleus σ (see [16]). Moreover, if $\mathbf{A}$ is totally ordered, then $\mathbf{G}$ can be chosen totally ordered. Another well-known result says that each totally ordered algebra from $\mathcal{ICRL}^{\mathcal{C}}$ satisfying $x \wedge y = x(x \to y)$ is an ordinal sum of totally ordered Wajsberg hoops (see [1]). Moreover, each Wajsberg hoop is a nuclear retraction of the negative cone of an Abelian ℓ-group (see [10]). In other words, for any Wajsberg hoop $\mathbf{A}$ there is an Abelian ℓ-group $\mathbf{G}$ and a nucleus γ such that $\mathbf{A}$ is isomorphic to the γ-retraction of the σ-contraction of $\mathbf{G}$ for the conucleus $\sigma(x) = x \wedge 0$. Similarly as before, if $\mathbf{A}$ is totally ordered, $\mathbf{G}$ can be chosen totally ordered as well.

Considering the above-mentioned results, it is natural to ask how far we can go with the connection to Abelian ℓ-groups. Can be each member of $\mathcal{ICRL}^{\mathcal{C}}$ described as a nuclear retraction of a conuclear contraction of an Abelian ℓ-group? In this paper we will identify within the class of finite totally ordered members of $\mathcal{ICRL}^{\mathcal{C}}$ those whose structure is related to totally ordered Abelian ℓ-groups. More precisely, let $\mathbf{A} \in \mathcal{ICRL}^{\mathcal{C}}$ be finite and totally ordered. We will characterize when $\mathbf{A}$ is a nuclear retraction of a conuclear contraction of a totally ordered Abelian ℓ-group $\mathbf{G}$. In Section 3, we will see that $\mathbf{A}$ can be obtained as a nuclear retraction of a conuclear contraction of $\mathbf{G}$ if, and only if, its ℓ-monoidal reduct is a homomorphic image of a totally ordered ℓ-monoid whose monoidal reduct is free. We call an ℓ-monoid satisfying this condition *regular* (see [6][1]).

2 Preliminaries

In this section we set a general notation used in this paper and recall necessary definitions of algebras used in the sequel. The set of natural numbers, integers, rational numbers, and reals are denoted respectively by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Let $f\colon A \to B$ be a mapping and $y \in B$. The symbol $f^{-1}(y)$ denotes the set $\{x \in A \mid f(x) = y\}$. If $\mathbb{A} \in \mathbb{R}^{m\times n}$ is a matrix then $\mathbb{A}\mathbb{N}^n = \{\mathbb{A}x \in \mathbb{R}^m \mid x \in \mathbb{N}^n\}$.

The commutative free monoid over the set of generators I will be denoted by $F_{\mathbb{N}}(I)$, i.e., $F_{\mathbb{N}}(I)$ consists of all functions from I to the set of natural numbers $\mathbb{N}$ whose supports are finite. In particular, if I is finite, then $F_{\mathbb{N}}(I) = \mathbb{N}^I$. Similarly we denote the free Abelian group $F_{\mathbb{Z}}(I)$ and the free $\mathbb{Q}$-vector space $F_{\mathbb{Q}}(I)$. The sets $F_{\mathbb{Z}}(I)$ and $F_{\mathbb{Q}}(I)$ consist of all functions from I to $\mathbb{Z}$ and $\mathbb{Q}$ respectively, whose supports are finite. Clearly, we have $F_{\mathbb{N}}(I) \subseteq F_{\mathbb{Z}}(I) \subseteq F_{\mathbb{Q}}(I)$. All the algebras $F_{\mathbb{N}}(I), F_{\mathbb{Z}}(I), F_{\mathbb{Q}}(I)$ are generated by the set of functions $\{\varepsilon_i \in \mathbb{N}^I \mid i \in I\}$ where

$$\varepsilon_i(x) = \begin{cases} 1 & \text{if } x = i\,, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{A}$ be an algebra. The congruence lattice of $\mathbf{A}$ is denoted by $\mathrm{Con}(\mathbf{A})$. Let $\theta \in \mathrm{Con}(\mathbf{A})$ and $a \in A$. The congruence class containing a with respect to θ is denoted by a/θ. Let $\mathbf{R} = (\mathbb{R}, +, -, \leq, 0)$ be the totally ordered additive ℓ-group of real numbers. The finite lexicographic product of n-copies of $\mathbf{R}$ is the totally ordered Abelian ℓ-group $\mathbf{R}^n_{\mathrm{lex}} = (\mathbb{R}^n, +, -, \leq_{\mathrm{lex}}, 0)$, where $\leq_{\mathrm{lex}}$ denotes the lexicographic order on $\mathbb{R}^n$ (see e.g. [7]).

2.1 Integral commutative residuated chains

All monoids in this paper are commutative. Thus by the word “monoid” we always mean a commutative monoid. A *lattice ordered monoid* (ℓ-monoid) is an algebra $\mathbf{S} = (S, \cdot, \wedge, \vee, 1)$, where $(S, \cdot, 1)$ is a monoid, $(S, \wedge, \vee)$ is a lattice and the identity $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ is valid in $\mathbf{S}$. Instead of $a \cdot b$ we write shortly ab. An ℓ-monoid $\mathbf{S} = (S, \cdot, \wedge, \vee, 1)$ is called *integral* if $s \leq 1$ for all $s \in S$. For us the most important subclass of ℓ-monoids will be the class of totally ordered ℓ-monoids which we call shortly *ordered monoids.*

It is well known that each finitely generated integral ordered monoid is dually well-ordered due to Dickson's Lemma. This result can be further extended to any dually well-ordered set of generators.

[1]The authors of [6] calls such an ℓ-monoid *formally integral.* We change the name because we use the word “integral” in a different meaning.

Theorem 2.1 ([14]) *An integral ordered monoid* $\mathbf{S}$ *which is generated by a dually well-ordered set is dually well-ordered as well.*

An ℓ-monoid $\mathbf{S} = (S, \cdot, \wedge, \vee, 1)$ is said to be *residuated* if for each $a, b \in S$ the inequality $ax \leq b$ has a maximum solution. This solution is called a *residuum* and is denoted $a \to b$. The enriched algebra $\mathbf{S} = (S, \cdot, \to, \wedge, \vee, 1)$ is called a *commutative residuated lattice* (CRL). A totally ordered CRL is referred to as a *commutative residuated chain* (CRC). We call a CRL, which is integral as an ℓ-monoid, an *integral CRL* (ICRL). Analogously ICRC stands for *integral CRC*.

It is well known that the class of ICRLs forms a congruence distributive variety $\mathcal{ICRL}$. The class of subdirect products of ICRCs forms a subvariety of $\mathcal{ICRL}$ denoted $\mathcal{ICRL}^{\mathcal{C}}$. The members of $\mathcal{ICRL}^{\mathcal{C}}$ are called *representable* ICRLs.

2.2 Nucleus and conucleus

Let $\mathbf{P}$ be a poset. A mapping $\gamma\colon P \to P$ is called *closure operator* if it is expanding, monotone and idempotent, i.e., $x \leq \gamma(x)$, $x \leq y$ implies $\gamma(x) \leq \gamma(y)$, and $\gamma(\gamma(x)) = \gamma(x)$ for all $x, y \in P$. Dually, an *interior operator* $\sigma\colon P \to P$ is a contracting ($\sigma(x) \leq x$), monotone and idempotent mapping on $\mathbf{P}$. The closure and interior operators γ and σ are completely determined by their images $P_\gamma = \gamma(P)$ and $P_\sigma = \sigma(P)$ (see [10, 16]), namely

$$\gamma(x) = \min\{a \in P_\gamma \mid x \leq a\}\,, \quad \sigma(x) = \max\{a \in P_\sigma \mid a \leq x\}\,.$$

A closure operator γ on a CRL $\mathbf{A}$ is called a *nucleus* if for all $x, y \in A$ we have $\gamma(x)\gamma(y) \leq \gamma(xy)$. Dually, an interior operator σ on $\mathbf{A}$ is said to be a *conucleus* if $\sigma(x)\sigma(y) \leq \sigma(xy)$ for all $x, y \in A$ and $\sigma(1) = 1$. For details on these notions see [8, 10, 16].

Let $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ be a CRC and γ a nucleus on $\mathbf{A}$. The algebra $\mathbf{A}_\gamma = (A_\gamma, \cdot_\gamma, \to, \wedge, \vee, \gamma(1))$, where $x \cdot_\gamma y = \gamma(xy)$, is called the *$\gamma$-retraction* of $\mathbf{A}$. Dually, if σ is a conucleus on $\mathbf{A}$ then the algebra $\mathbf{A}_\sigma = (A_\sigma, \cdot, \to_\sigma, \wedge, \vee, 1)$, where $x \to_\sigma y = \sigma(x \to y)$, is called the *$\sigma$-contraction* of $\mathbf{A}$.

Theorem 2.2 ([8]) *Let* $\mathbf{A}, \mathbf{B}$ *be CRCs,* γ *a nucleus on* $\mathbf{A}$*, and* σ *a conucleus on* $\mathbf{B}$*. Then the γ-retraction* $\mathbf{A}_\gamma$ *and the σ-contraction* $\mathbf{B}_\sigma$ *are CRCs.*

It is not difficult to prove that each nucleus γ satisfies $\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$ and each conucleus σ satisfies $\sigma(\sigma(x)\sigma(y)) = \sigma(x)\sigma(y)$. Due to these facts we have the following lemma.

Lemma 2.3 *Let* $\mathbf{A}, \mathbf{B}$ *be CRCs,* γ *a nucleus on* $\mathbf{A}$*, and* σ *a conucleus on* $\mathbf{B}$*. Then* $\mathbf{B}_\sigma$ *forms a sub-ℓ-monoid of* $\mathbf{B}$ *and* γ *is an ℓ-monoidal homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}_\gamma$*.*

Although a nucleus γ on a CRC $\mathbf{A}$ is an ℓ-monoidal homomorphism, it need not preserve the residuum. Nevertheless, we have at least the following lemma.

Lemma 2.4 *Let* $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ *be a CRC and* γ *a nucleus on* $\mathbf{A}$*. If* $a \in A$ *and* $b \in A_\gamma$*, then* $\gamma(a \to b) = a \to b = \gamma(a) \to b = \gamma(a) \to \gamma(b)$*. In particular,* $a \to b \in A_\gamma$*.*

PROOF: Clearly, $\gamma(a) \to \gamma(b) = \gamma(a) \to b$ since we assume that $b \in A_\gamma$. Further, $a \to b \leq \gamma(a \to b)$. Conversely, since $a(a \to b) \leq b$, we have $\gamma(a)\gamma(a \to b) \leq \gamma(a(a \to b)) \leq \gamma(b) = b$

showing that $\gamma(a \to b) \le \gamma(a) \to b$. Moreover, $\gamma(a) \to b \le a \to b$ since $\gamma(a) \ge a$. All together we have

$$a \to b \le \gamma(a \to b) \le \gamma(a) \to b \le a \to b\,.$$

□

2.3 Cones and orders

Further, we have to recall several facts on orders on $F_{\mathbb{N}}(I)$, $F_{\mathbb{Z}}(I)$, and $F_{\mathbb{Q}}(I)$. Let $\mathbf{V}$ be a $\mathbb{Q}$-vector space. A subset $C \subseteq V$ is called a *convex cone* if for all non-negative rational numbers α, β we have $\alpha C + \beta C \subseteq C$. In addition, if C contains no proper subspace then C is said to be *pointed*. A pointed cone C is said to be a *T-cone* if $V = C \cup -C$. T-cones are in fact maximal (with respect to inclusion) pointed cones on $\mathbf{V}$ and each pointed cone C can be extended to a T-cone by Zorn's Lemma.

Each total order $\le$ on $\mathbf{V}$, which makes $\mathbf{V}$ a totally ordered Abelian ℓ-group, can be characterized by means of a T-cone. Namely, we can assign to $\le$ a T-cone $C_{\le} = \{x \in V \mid 0 \le x\}$. Conversely, if C is a T-cone, then the relation defined by setting $x \le_C y$ iff $y - x \in C$ is a total order making $\mathbf{V}$ a totally ordered Abelian ℓ-group.

The total orders on vector spaces are important for us since each total order on $F_{\mathbb{N}}(I)$, which makes $F_{\mathbb{N}}(I)$ an ordered monoid, uniquely determines a total order on $F_{\mathbb{Q}}(I)$. Namely, we have the following proposition (for the proof see e.g. [6]).

Proposition 2.5 *Each total order on $F_{\mathbb{N}}(I)$ making $F_{\mathbb{N}}(I)$ an ordered monoid can be uniquely extended to a total order on $F_{\mathbb{Q}}(I)$ (resp. $F_{\mathbb{Z}}(I)$) making $F_{\mathbb{Q}}(I)$ (resp. $F_{\mathbb{Z}}(I)$) a totally ordered Abelian ℓ-group.*

If we restrict ourselves to finite dimensional $\mathbb{Q}$-vector spaces, then we have even better characterizations of total orders. Let $\mathbb{A} \in \mathbb{R}^{m\times n}$ be a matrix. The ith-row of $\mathbb{A}$ is denoted $\mathbb{A}_i \in \mathbb{R}^m$. Let $x, y \in \mathbb{R}^m$. Then $\|x\|$ denotes the usual Euclidean norm of x and $x \perp y$ expresses that x and y are perpendicular.

Definition 2.6 A matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$ is called *canonical* if the following conditions are satisfied:

1. $\mathbb{A}_i \perp \mathbb{A}_j$ for $i \neq j$,
2. $\|\mathbb{A}_i\| = 1$,
3. $\{x \in \mathbb{Q}^n \mid \mathbb{A}x = 0\} = \{0\}$.

Each canonical matrix defines a total order on $\mathbb{Q}^n$ making it a totally ordered Abelian ℓ-group by setting $x \le y$ iff $\mathbb{A}x \le_{\text{lex}} \mathbb{A}y$. Moreover, each total order on $\mathbb{Q}^n$ is of this type (see [12, 17]).

Theorem 2.7 *Let $\le$ be a total order on $\mathbb{Q}^n$ making it a totally ordered Abelian ℓ-group. Then there is a uniquely determined canonical matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, $m \le n$, such that $x \le y$ iff $\mathbb{A}x \le_{\text{lex}} \mathbb{A}y$.*

2.4 Regular ordered monoids

Now we introduce the important class of ordered monoids whose structure can be described by means of a totally ordered Abelian ℓ-group.

Definition 2.8 An ordered monoid $\mathbf{S} = (S, \cdot, \leq_S, 1)$ is called *regular* if there is a set I, a surjective monoidal homomorphism $\phi\colon F_{\mathbb{N}}(I) \to S$ and a total order $\leq$ on $F_{\mathbb{N}}(I)$ making $F_{\mathbb{N}}(I)$ an ordered monoid such that $x \leq y$ implies $\phi(x) \leq_S \phi(y)$, i.e., $\mathbf{S}$ is a homomorphic image (as an ℓ-monoid) of the ordered monoid $(F_{\mathbb{N}}(I), +, \leq, 0)$.

Observe that regularity does not seem to be a very strict condition since each monoid is a homomorphic image of a free monoid. We only require that at least one of such free monoids can be totally ordered so that the homomorphism becomes order-preserving. The fact that an ordered monoid is regular can be recognized from a certain convex cone in $F_{\mathbb{Q}}(I)$.

Definition 2.9 Let $\mathbf{S} = (S, \cdot, \leq_S, 1)$ be an ordered monoid and $\phi : F_{\mathbb{N}}(I) \to S$ a monoidal homomorphism. Then

$$D(\phi) = \{y - x \in F_{\mathbb{Z}}(I) \mid x, y \in F_{\mathbb{N}}(I) \text{ and } \phi(x) <_S \phi(y)\}\,.$$

The convex cone in $F_{\mathbb{Q}}(I)$ generated by $D(\phi)$ will be denoted $C(\phi)$.

The following criteria of regularity were proved in [6].

Theorem 2.10 ([6]) *For any ordered monoid* $\mathbf{S}$*, the following are equivalent:*

1. $\mathbf{S}$ *is regular,*
2. *for some surjective monoidal homomorphism* $\phi\colon F_{\mathbb{N}}(I) \to S$, $C(\phi) \subseteq F_{\mathbb{Q}}(I)$ *is pointed,*
3. *for all monoidal homomorphisms* $\phi\colon F_{\mathbb{N}}(I) \to S$, $C(\phi) \subseteq F_{\mathbb{Q}}(I)$ *is pointed.*

Let us also recall the following facts which we will need in the sequel.

Proposition 2.11 ([6]) *Let* $\mathbf{S}$ *be a regular ordered monoid. Then we have the following:*

1. *Each sub-*ℓ*-monoid of* $\mathbf{S}$ *is regular as well.*
2. *If* $\phi\colon F_{\mathbb{N}}(I) \to S$ *is a monoidal homomorphism, then there is a total order on* $F_{\mathbb{N}}(I)$ *making* $F_{\mathbb{N}}(I)$ *an ordered monoid and* ϕ *an* ℓ*-monoidal homomorphism.*

3 Structure of regular ordered monoids and ICRCs

We start with the statement establishing the connection between regular ordered monoids and totally ordered Abelian ℓ-groups. It is in fact an easy consequence of results published in [6].

Theorem 3.1 *An ordered monoid* $\mathbf{S}$ *is a quotient of a sub-*ℓ*-monoid of a totally ordered Abelian* ℓ*-group* $\mathbf{G}$ *if, and only if,* $\mathbf{S}$ *is regular.*

PROOF: Let $\mathbf{S}$ be a regular ordered monoid. This means that $\mathbf{S}$ is a quotient (as an ℓ-monoid) of $(F_{\mathbb{N}}(I), +, \leq, 0)$. Since the order on $F_{\mathbb{N}}(I)$ can be extended to $F_{\mathbb{Z}}(I)$ by Proposition 2.5, $(F_{\mathbb{N}}(I), +, \leq, 0)$ is a sub-ℓ-monoid of the totally ordered Abelian ℓ-group $(F_{\mathbb{Z}}(I), +, \leq, 0)$.

Conversely, each totally ordered Abelian ℓ-group $\mathbf{G}$ is regular by [6, Corollary 4.5]. Since regularity is preserved under taking sub-ℓ-monoids by Proposition 2.11 and quotients by its definition, we are done. □

Observe that the latter theorem tells us quite a lot about the structure of regular ordered monoids since each totally ordered Abelian ℓ-group can be embedded into a full Hahn group by Hahn's Embedding Theorem, i.e., an ℓ-group of functions from a totally ordered set I to $\mathbb{R}$ whose supports are either empty or dually well-ordered (for details see [7, 11]). For finitely generated ordered monoids we can in fact obtain a better description. The following theorem can be viewed as a generalization of [6, Proposition 7.2].

Theorem 3.2 *Let $\mathbf{S} = (S, \cdot, \leq_S, 1)$ be an n-generated regular ordered monoid. Then there is a canonical matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, $m \leq n$, such that $\mathbf{S}$ is a homomorphic image of $\mathbf{A} = (\mathbb{A}\mathbb{N}^n, +, \leq_{\text{lex}}, 0)$.*

If, in addition, $\mathbf{S}$ is integral then $\mathbb{A}\mathbb{N}^n$ is a subset of the negative cone of $\mathbf{R}^m_{\text{lex}}$, i.e., $\mathbf{A}$ is integral as well.

PROOF: Let $\phi\colon \mathbb{N}^n \to S$ be the monoidal epimorphism from the free n-generated monoid to $\mathbf{S}$. We may assume without any loss of generality that $\phi^{-1}(1) = \{0\}$. Since $\mathbf{S}$ is regular, it follows from Theorem 2.10 that the cone $C(\phi)$ is pointed. Let C be a maximal pointed cone containing $C(\phi)$. Then C is a T-cone. Let $\leq_C$ be its corresponding total order on $\mathbb{Q}^n$ hence also on $\mathbb{N}^n$ (see Section 2.3). Then by Theorem 2.7 there is a canonical matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, $m \leq n$, such that $x \leq_C y$ iff $\mathbb{A}x \leq_{\text{lex}} \mathbb{A}y$. Define $\psi\colon \mathbb{A}\mathbb{N}^n \to S$ by $\psi(\mathbb{A}x) = \phi(x)$. The mapping ψ is a monoidal morphism since $\phi(0) = 1$ and

$$\psi(\mathbb{A}x + \mathbb{A}y) = \psi(\mathbb{A}(x + y)) = \phi(x + y) = \phi(x) + \phi(y) = \psi(\mathbb{A}x) + \psi(\mathbb{A}y)\,.$$

Finally, we have to check that ψ is order-preserving, i.e., $\mathbb{A}x \leq_{\text{lex}} \mathbb{A}y$ implies $\psi(\mathbb{A}x) \leq_S \psi(\mathbb{A}y)$. Assume that $\psi(\mathbb{A}x) >_S \psi(\mathbb{A}y)$. Then $\phi(x) >_S \phi(y)$ and $x \neq y$. By definition $x - y \in D(\phi) \subseteq C(\phi) \subseteq C$. Thus $x >_C y$, i.e., $\mathbb{A}x >_{\text{lex}} \mathbb{A}y$.

Let $\varepsilon_1, \dots, \varepsilon_n$ be the generators of the free monoid on $\mathbb{N}^n$. To see the second claim note that if $\mathbf{S}$ is integral then $\phi(\varepsilon_i) < \phi(0)$ for all $i = 1, \dots, n$. Thus $-\varepsilon_i \in D(\phi)$. Consequently, for all $x \in \mathbb{N}^n$ we have $x \leq_C 0$ since

$$0 - x = \sum_{i=1}^{n} k_i(-\varepsilon_i) \in C(\phi) \subseteq C\,,$$

for some $k_i \in \mathbb{N}$. Thus $\mathbb{A}x \leq_{\text{lex}} 0$. □

For finite ICRCs we can obtain a characterization of regularity by means of conuclei and nuclei. We start with the following lemma.

Lemma 3.3 *Let $\mathbf{A}$, $\mathbf{B}$ be ICRCs and $\phi\colon A \to B$ an ℓ-monoidal surjective homomorphism such that for each $y \in B$ the set $\phi^{-1}(y)$ has a maximum. Then the mapping $\gamma\colon A \to A$ defined by*

$$\gamma(x) = \max \phi^{-1}(\phi(x))\,,$$

is a nucleus on $\mathbf{A}$ *and* $\mathbf{A}_\gamma$ *is isomorphic to* $\mathbf{B}$*.*

PROOF: First, we prove that γ is a nucleus. Let $x, y \in A$. Since $x \in \phi^{-1}(\phi(x))$, we have $x \leq \max \phi^{-1}(\phi(x)) = \gamma(x)$. Suppose that $x \leq y$. Since ϕ is order-preserving, we get $\phi(x) \leq \phi(y)$. Consequently, $\gamma(x) \leq \gamma(y)$. Clearly, $\gamma(\gamma(x)) = \gamma(x)$ since max is idempotent. Finally, observe that $\phi(\gamma(x)) = \phi(x)$. Thus

$$\phi(\gamma(x)\gamma(y)) = \phi(\gamma(x))\phi(\gamma(y)) = \phi(x)\phi(y) = \phi(xy)\,.$$

Consequently, $\gamma(x)\gamma(y) \in \phi^{-1}(\phi(xy))$, i.e., $\gamma(x)\gamma(y) \leq \gamma(xy)$. Hence γ is a nucleus.

Second, we will show that $\mathbf{A}_\gamma \cong \mathbf{B}$. Let $h : A_\gamma \to B$ be a mapping defined by $h(\gamma(x)) = \phi(x)$. The mapping h is a monoidal morphism since for any $x, y \in A_\gamma$ (i.e., $x = \gamma(x)$ and $y = \gamma(y)$) we have

$$h(x \cdot_\gamma y) = h(\gamma(xy)) = \phi(xy) = \phi(x)\phi(y) = h(\gamma(x))h(\gamma(y)) = h(x)h(y)\,.$$

It is clearly onto since ϕ is onto. Let $\gamma(x) < \gamma(y)$. Then $\max \phi^{-1}(\phi(x)) < \max \phi^{-1}(\phi(y))$. Thus $\phi(x) < \phi(y)$, i.e., h is one-to-one and order-preserving. Thus $\mathbf{A}_\gamma$ and $\mathbf{B}$ are isomorphic as ℓ-monoids. Since the residuum is fully determined by the monoidal operation and the order, they are isomorphic also as ICRCs. □

Now we can state the main theorem of this paper classifying those finite ICRCs which can be obtained as a nuclear retraction of a conuclear contraction of a totally ordered Abelian ℓ-group. The theorem is slightly more general since it holds not only for finite ICRCs but also for ICRCs whose monoidal reduct is generated by a dually well-ordered set.

Theorem 3.4 *Let* $\mathbf{A}$ *be an ICRC whose monoidal reduct is generated by a dually well-ordered set. Then the* ℓ*-monoidal reduct of* $\mathbf{A}$ *is regular if, and only if,* $\mathbf{A}$ *is isomorphic to* $(\mathbf{G}_\sigma)_\gamma$ *for a totally ordered Abelian* ℓ*-group* $\mathbf{G}$*, a conucleus* σ*, and a nucleus* γ*. Moreover, the right-to-left implication holds for any CRC* $\mathbf{A}$*.*

PROOF: If $\mathbf{A} \cong (\mathbf{G}_\sigma)_\gamma$ then the ℓ-monoidal reduct of $\mathbf{A}$ is a quotient of a sub-ℓ-monoid of $\mathbf{G}$ by Lemma 2.3. Thus the ℓ-monoidal reduct of $\mathbf{A}$ is regular by Theorem 3.1.

Conversely, assume that the ℓ-monoidal reduct of $\mathbf{A}$ is regular. Since the ℓ-monoidal reduct of $\mathbf{A}$ is generated by a dually well-ordered set $I \subseteq A \setminus \{1\}$, $\mathbf{A}$ is a homomorphic image (as a monoid) of $F_{\mathbb{N}}(I)$. For each $i \in I$ the corresponding surjective monoidal homomorphism $\phi : F_{\mathbb{N}}(I) \to A$ maps ε_i to i. Moreover, there is a total order $\leq$ on $F_{\mathbb{N}}(I)$ making $F_{\mathbb{N}}(I)$ an ordered monoid and ϕ an ℓ-monoidal homomorphism by Proposition 2.11. Observe that the set $E = \{\varepsilon_i \in F_{\mathbb{N}}(I) \mid i \in I\}$ is dually well-ordered. Indeed, it follows from the fact that ϕ restricted to E is an order-preserving bijection between E and I. Consequently, $F_{\mathbb{N}}(I)$ is dually well-ordered as well by Theorem 2.1. Thus $F_{\mathbb{N}}(I)$ is residuated, i.e., it forms an ICRC. By Lemma 3.3 the mapping $\gamma\colon F_{\mathbb{N}}(I) \to F_{\mathbb{N}}(I)$ defined by

$$\gamma(x) = \max \phi^{-1}(\phi(x))\,,$$

is a nucleus and $\mathbf{A}$ is isomorphic to $(F_{\mathbb{N}}(I))_\gamma$. The mapping γ is well-defined since $F_{\mathbb{N}}(I)$ is dually well-ordered.

Since $\leq$ can be extended to a total order on $F_{\mathbb{Z}}(I)$ making it a totally ordered Abelian ℓ-group by Proposition 2.5, we can view $F_{\mathbb{N}}(I)$ as a sub-ℓ-monoid of $F_{\mathbb{Z}}(I)$. Define a mapping $\sigma\colon F_{\mathbb{Z}}(I) \to F_{\mathbb{Z}}(I)$ as follows:

$$\sigma(a) = \max\{x \in F_{\mathbb{N}}(I) \mid x \leq a\}\,.$$

The mapping σ is again well-defined because $F_{\mathbb{N}}(I)$ is dually well-ordered. It is easy to check that σ is a conucleus and $F_{\mathbb{N}}(I) = (F_{\mathbb{Z}}(I))_\sigma$. □

Using Theorem 3.2, we can describe the structure of a finite regular ICRC $\mathbf{A}$. Let $\mathbb{A} \in \mathbb{R}^{m\times n}$ be a matrix. Note that if $(\mathbb{A}\mathbb{N}^n, +, \leq_{\text{lex}}, 0)$ is integral (i.e., a sub-ℓ-monoid of the negative cone of $\mathbf{R}^m_{\text{lex}}$), then $(\mathbb{A}\mathbb{N}^n, +, \leq_{\text{lex}}, 0)$ is dually well-ordered by Theorem 2.1. Consequently, $(\mathbb{A}\mathbb{N}^n, +, \to, \leq_{\text{lex}}, 0)$ forms an ICRC. Then the following theorem follows from Lemma 3.3.

Theorem 3.5 *Let $\mathbf{A}$ be an ICRC whose ℓ-monoidal reduct is n-generated and regular. Further, let $\mathbb{A} \in \mathbb{R}^{m\times n}$, $m \leq n$, be the canonical matrix from Theorem 3.2 and $\phi\colon \mathbb{A}\mathbb{N}^n \to A$ the corresponding ℓ-monoidal homomorphism. Then $\mathbf{A}$ is isomorphic to the γ-retraction of $(\mathbb{A}\mathbb{N}^n, +, \to, \leq_{\text{lex}}, 0)$ for the nucleus $\gamma\colon \mathbb{A}\mathbb{N}^n \to \mathbb{A}\mathbb{N}^n$ defined by $\gamma(x) = \max \phi^{-1}(\phi(x))$.*

4 Known classes of regular ICRCs

It is quite natural to ask how wide is the class of regular ICRCs since we would like to know where Theorems 3.4 and 3.5 are applicable. In [6] there is a hint saying that this class is quite wide since it is not very easy to come up with an example of an ordered monoid which is not regular. Another way of testing the size of this class, is to look at the well-known classes of ICRCs from the literature and discuss whether they are regular or not.

Likely the best understood subclass of representable ICRLs is the class of *basic hoops*, i.e., representable ICRLs satisfying $x \wedge y = x(x \to y)$. As we mentioned in the introduction each totally ordered basic hoop is an ordinal sum of totally ordered Wajsberg hoops (see [1]). Further, each totally ordered Wajsberg hoop is either the negative cone of a totally ordered Abelian ℓ-group $\mathbf{G}$ or an interval of the form $[a, 0]$ for some $a \in G$ (see [2]). Consequently, it follows from Theorem 3.1 that the ℓ-monoidal reduct of any totally ordered Wajsberg hoop is regular. Moreover, we proved in [13] that the ordinal sum preserves regularity, i.e., the ordinal sum $\bigoplus_{i\in I} \mathbf{A}_i$ of a family $\{\mathbf{A}_i \mid i \in I\}$ of ICRCs is regular if, and only if, all $\mathbf{A}_i$'s are regular. Thus we obtain the following theorem.

Theorem 4.1 *The ℓ-monoidal reduct of any totally ordered basic hoop is regular.*

The ordinal sum is one of the most known construction methods for ICRCs. As we mentioned, it preserves regularity. Another well-known construction method is the so-called disconnected rotation producing from an ICRC an involutive ICRC. This construction for ICRCs was introduced in [15] and independently also in [9]. Let $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ be an ICRC. The *disconnected rotation* $\mathbf{A}^* = (A^*, *, \to_*, \wedge_*, \vee_*, 1', 1)$ of $\mathbf{A}$ is an algebra whose universe is $A^* = A \cup A'$, where $A' = \{a' \mid a \in A\}$ is a disjoint copy of A. Let $\leq$ be the order on $\mathbf{A}$. Consider A' with the inverse order and extend the order to A^* by letting $a' < b$ for all $a, b \in A$. Then $\wedge_*$ and $\vee_*$ are just minimum and maximum with respect to this extended order. Let $x, y \in A$. The operations $*$ and $\to_*$ are defined as follows:

$$x * y = xy\,, \qquad x * y' = y' * x = (x \to y)'\,, \qquad x' * y' = 1'\,,$$

$$x \to_* y = x \to y\,, \qquad x \to_* y' = (xy)'\,, \qquad x' \to_* y = 1\,, \qquad x' \to_* y' = y \to x\,.$$

Proposition 4.2 ([15]) *Let* $\mathbf{A}$ *be an ICRC. Then the disconnected rotation* $\mathbf{A}^*$ *of* $\mathbf{A}$ *is an involutive bounded ICRC, i.e., an ICRC with a bottom element* $1'$ *satisfying the double negation law* $\neg\neg x = (x \to_* 1') \to_* 1' = x$.

Now we prove that the disconnected rotation preserves regularity for a certain class of ICRCs, in particular for finite ICRCs. Let $\mathbf{Z}$ be the totally ordered Abelian ℓ-group of integers.

Theorem 4.3 *Let* $\mathbf{A} = (A, \cdot_\gamma, \to, \wedge, \vee, 0)$ *be an ICRC isomorphic to* $(\mathbf{G}_\sigma)_\gamma$ *for a totally ordered Abelian* ℓ*-group* $\mathbf{G} = (G, +, -, \wedge, \vee, 0)$*, a conucleus* σ*, and a nucleus* γ*. Then* $\mathbf{A}^*$ *is regular.*

PROOF: Observe that G_σ is a subset of the negative cone of G otherwise $\mathbf{A}$ would not be integral. Consider the lexicographic product $\mathbf{G}^*$ of $\mathbf{Z}$ and $\mathbf{G}$, i.e., $G^* = \mathbb{Z} \times G$ ordered lexicographically. We will prove that $\mathbf{A}^*$ is a homomorphic image of the sub-ℓ-monoid $\mathbf{M}^*$ of $\mathbf{G}^*$ generated by the set

$$C^* = \{\langle 0, x\rangle \in G^* \mid x \in G_\sigma\} \cup \{\langle -1, a-b\rangle \in G^* \mid a \geq b,\ a \in G_\sigma,\ b \in (G_\sigma)_\gamma\}\,.$$

We in fact define two homomorphisms whose composition gives the desired result. First, if we identify all the elements less than or equal to $\langle -1, 0\rangle$, we obtain a homomorphic image $\mathbf{M}_1^*$ of $\mathbf{M}^*$. In fact the universe of $\mathbf{M}_1^*$ is C^*. Now we define a mapping $\phi\colon M_1^* \to A^*$ as follows:

$$\phi(z,x) = \begin{cases} \gamma(x) & \text{if } z = 0, \\ ((\gamma\sigma)(-x))' & \text{if } z = -1. \end{cases}$$

The symbol $\gamma\sigma$ stands for the composition of mappings σ and γ. We will check that ϕ is a monoidal homomorphism. Let $\langle z_1, x_1\rangle, \langle z_2, x_2\rangle \in M_1^*$. There are several cases. If $z_1 = z_2 = 0$ then

$$\begin{aligned}\phi(z_1+z_2, x_1+x_2) = \phi(0, x_1+x_2) = \gamma(x_1+x_2) = \\ = \gamma(\gamma(x_1)+\gamma(x_2)) = \gamma(x_1)\cdot_\gamma \gamma(x_2) = \phi(z_1,x_1) * \phi(z_2,x_2)\,.\end{aligned}$$

If $z_1 = -1$ and $z_2 = 0$ then $x_1 = a - b$ for some $a \in G_\sigma$ and $b \in (G_\sigma)_\gamma$. Thus we have

$$\phi(z_1+z_2, x_1+x_2) = \phi(-1, a-b+x_2) = ((\gamma\sigma)(b-a-x_2))'\,.$$

Note that the residuum $\to$ in $\mathbf{A}$ is just the restriction of the residuum in $\mathbf{G}_\sigma$. Hence we use the same symbol also to denote the residuum in $\mathbf{G}_\sigma$. Thus we have $x \to y = \sigma(y-x)$ for $x, y \in G_\sigma$. Consequently,

$$((\gamma\sigma)(b-a-x_2))' = (\gamma((x_2+a) \to b))' = (\gamma(x_2 \to (a \to b)))'\,.$$

The second equality follows since $xy \to z = x \to (y \to z)$ holds in any ICRL. Since $b \in (G_\sigma)_\gamma$, we get $a \to b \in (G_\sigma)_\gamma$ by Lemma 2.4. Thus using Lemma 2.4 once again we obtain

$$\begin{aligned}(\gamma(x_2 \to (a \to b)))' = (\gamma(x_2) \to \gamma(a \to b))' = \\ = \gamma(x_2) * \gamma(a \to b)' = \gamma(x_2) * ((\gamma\sigma)(b-a))' = \phi(z_1,x_1) * \phi(z_2,x_2)\,.\end{aligned}$$

If $z_1 = z_2 = -1$ then we have

$$\phi(z_1 + z_2, x_1 + x_2) = \phi(-1, 0) = ((\gamma\sigma)(0))' = 0' = \\ = ((\gamma\sigma)(-x_1))' * ((\gamma\sigma)(-x_2))' = \phi(z_1, x_1) * \phi(z_2, x_2)\,.$$

Next we check that ϕ is surjective. Let $y \in A^*$. Clearly, if $y \in A$ then $y \in (G_\sigma)_\gamma$. Thus $\phi(0, y) = \gamma(y) = y$. If $y \notin A$ then $y = s'$ for some $s \in (G_\sigma)_\gamma$. Consequently, we have

$$\phi(-1, -s) = ((\gamma\sigma)(s))' = s' = y\,.$$

Finally, we have to show that ϕ is order-preserving. Let $\langle z_1, x_1\rangle, \langle z_2, x_2\rangle \in M_1^*$ such that $\langle z_1, x_1\rangle \leq_{\mathrm{lex}} \langle z_2, x_2\rangle$. The case for $z_1 = z_2 = 0$ is trivial. Also the case when $z_1 = -1$ and $z_2 = 0$ is obvious because $\phi(z_1, x_1) \in A'$ and $\phi(z_2, x_2) \in A$. Thus suppose that $z_1 = z_2 = -1$. Then $x_1 \leq x_2$, i.e., $-x_2 \leq -x_1$. Consequently, we obtain $(\gamma\sigma)(-x_2) \leq (\gamma\sigma)(-x_1)$. Hence

$$\phi(z_1, x_1) = ((\gamma\sigma)(-x_1))' \leq ((\gamma\sigma)(-x_2))' = \phi(z_2, x_2)\,.$$

Now, the regularity of $\mathbf{A}^*$ follows from Theorem 3.1. □

Corollary 4.4 *Let* $\mathbf{A}$ *be an ICRC whose monoidal reduct is generated by a dually well-ordered set. Then the* ℓ*-monoidal reduct of* $\mathbf{A}^*$ *is regular if, and only if, the* ℓ*-monoidal reduct of* $\mathbf{A}$ *is regular.*

PROOF: The left-to-right direction is obvious since $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{A}^*$ and regular ordered monoids are closed under taking subalgebras by Proposition 2.11.

Conversely, assume that the ℓ-monoidal reduct of $\mathbf{A}$ is regular. Then by Theorem 3.4 there is a totally ordered Abelian ℓ-group $\mathbf{G}$, a conucleus σ on $\mathbf{G}$ and a nucleus γ on $\mathbf{G}_\sigma$ such that $\mathbf{A} \cong (\mathbf{G}_\sigma)_\gamma$. Thus the statement follows by Theorem 4.3. □

5 The variety generated by regular ICRCs

Another interesting question concerning the regular ICRCs is whether they generate the variety $\mathcal{ICRL^C}$. Let us denote the variety generated by ICRCs whose ℓ-monoidal reduct is regular by $\mathcal{V}$. We will show that $\mathcal{V}$ is strictly smaller than $\mathcal{ICRL^C}$. We will start by the following example of a non-regular ordered monoid.

Example 5.1 ([6]) Let $a, b, c, d \in \mathbb{N}$. Then $\langle a, b, c\rangle$ denotes the sub-ℓ-monoid of $(\mathbb{N}, +, \leq, 0)$ generated by $\{a, b, c\}$. Moreover, $\langle a, b, c\rangle/d$ denotes its quotient where all elements greater than or equal to d are identified to one element denoted ∞.

Let $S = \{32^*\} \cup \langle 9, 12, 16\rangle/30$ denote the ordered monoid obtained from $\langle 9, 12, 16\rangle/30$ by adding one additional element, denoted by 32^*. This element satisfies $16 + 16 = 32^*$, $32^* + z = \infty$ for $z \neq 0$, and the whole monoid is to be ordered as follows:

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty\,.$$

All the relations that do not involve 32^* are as in $\langle 9, 12, 16\rangle/30$. This ordered monoid is not regular (for the proof see [6]).

Let us consider the ordered monoid $\mathbf{S} = (S, +, \leq, 0)$ from Example 5.1 which is not regular. We will construct from this ordered monoid an ICRC $\mathbf{A} = (A, +, \rightarrow, \wedge, \vee, 0)$. Let $A = S$ with the reverse order, i.e., the elements of A are ordered as follows:

$$0 > 9 > 12 > 16 > 18 > 21 > 24 > 25 > 27 > 28 > 32^* > \infty\,.$$

It is clear that a residuum exists since A is finite. Thus $\mathbf{A}$ is an ICRC whose ℓ-monoidal reduct is not regular. Now let us introduce the following identity in the language of ICRLs:

$$(x_1z_1 \rightarrow y_1z_2) \vee (x_2z_2 \rightarrow y_2z_1) \vee (y_1y_2 \rightarrow x_1x_2) = 1\,. \qquad (1)$$

This identity is not valid in $\mathbf{A}$. Indeed, let

$$x_1 = 16\,, \quad y_1 = 18\,, \quad z_1 = 16\,,$$
$$x_2 = 12\,, \quad y_2 = 9\,, \quad z_2 = 12\,.$$

Then we get the following:

$$\begin{aligned} x_1 + z_1 \rightarrow y_1 + z_2 &= 32^* \rightarrow \infty = 9\,, \\ x_2 + z_2 \rightarrow y_2 + z_1 &= 24 \rightarrow 25 = 9\,, \\ y_1 + y_2 \rightarrow x_1 + x_2 &= 27 \rightarrow 28 = 9\,. \end{aligned}$$

Thus

$$(x_1 + z_1 \rightarrow y_1 + z_2) \vee (x_2 + z_2 \rightarrow y_2 + z_1) \vee (y_1 + y_2 \rightarrow x_1 + x_2) = 9 \neq 0\,.$$

On the other hand we can prove that the identity (1) is valid in each ICRC whose ℓ-monoidal reduct is regular.

Proposition 5.2 *The identity (1) is valid in each ICRC whose ℓ-monoidal reduct is regular.*

PROOF: Assume that there is an ICRC $\mathbf{L} = (L, \cdot, \rightarrow, \wedge, \vee, 1)$ whose ℓ-monoidal reduct is regular and (1) is not valid in $\mathbf{L}$. Then there are elements $a_1, a_2, b_1, b_2, c_1, c_2 \in L$ such that $a_1c_1 > b_1c_2$, $a_2c_2 > b_2c_1$, and $b_1b_2 > a_1a_2$. Since $(L, \cdot, \wedge, \vee, 1)$ is regular, there is a set I, a total order $\leq$ on $F_{\mathbb{N}}(I)$ making it an ordered monoid, and a surjective order-preserving homomorphism $\phi\colon F_{\mathbb{N}}(I) \rightarrow L$. Let $a_1', a_2', b_1', b_2', c_1', c_2' \in F_{\mathbb{N}}(I)$ be arbitrary preimages of $a_1, a_2, b_1, b_2, c_1, c_2$ respectively. Then

$$a_1' + c_1' > b_1' + c_2'\,, \qquad a_2' + c_2' > b_2' + c_1'\,, \qquad b_1' + b_2' > a_1' + a_2'\,,$$

since ϕ is order-preserving. Adding the first two inequalities we get

$$a_1' + c_1' + a_2' + c_2' > b_1' + c_2' + b_2' + c_1'\,.$$

As $F_{\mathbb{N}}(I)$ is cancellative, we obtain $a_1' + a_2' > b_1' + b_2'$ (a contradiction with the third inequality above). □

Corollary 5.3 *The variety $\mathcal{ICRL}^{\mathcal{C}}$ is not generated by its members whose ℓ-monoidal reduct is regular.*

Problem 1 Find an axiomatization for the variety $\mathcal{V}$ generated by ICRCs whose ℓ-monoidal reduct is regular.

Another very interesting question concerns a generalization of our results from Section 3.

Problem 2 Is it possible to generalize Theorem 3.4 for any ICRC (not only those whose ℓ-monoidal reduct is generated by a dually well-ordered set)?

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