CONTENTS

Chapter I: Algebraic Semantics

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1 Substructural logics

Fuzzy logics were originally motivated by semantical considerations as logics whose intended set of truth values is the real unit interval $[0, 1]$. Later this was relaxed and $[0, 1]$ was replaced by an arbitrary linearly ordered set. Thus by a fuzzy logic we mean a logic complete with respect to a class of linearly ordered algebras. Since the notion of a fuzzy logic has not been precisely established, we will call such a logic *semilinear*. The term semilinear comes from the fact that the algebraic semantics for such a logic contains also algebras which are not linearly ordered but their building blocks (subdirectly irreducible algebras) are linearly ordered.

In order to discuss an algebraic semantics for semilinear logics, we have to specify precisely which logics we are going to study. We will put semilinear logics into a framework of substructural logics because most of the semilinear logics, studied so far, belong to the hierarchy of substructural logics. Note that in Chapter II semilinear logics are studied in a more general framework of *weakly implicative logics*. To keep the text within a reasonable size, we have to make several design choices. First, we will restrict ourselves only to logics in the full language of substructural logics. Thus we will not mention results on various fragments. Second, as a base substructural logic we consider the *full Lambek calculus* FL. This logic was defined already in the previous chapter as a weakly implicative logic and its Hilbert style calculus was given. Here we present FL by means of a Gentzen sequent system (see also Chapter IV). Then semilinear logics appear as axiomatic extensions of FL.

Given a set Q, we denote its powerset by $\mathcal{P}(Q)$. Recall that a *consequence relation* on a set Q is a binary relation $\vdash \subseteq \mathcal{P}(Q) \times Q$ such that for every $X \cup Y \cup \{x, z\} \subseteq Q$ we have

- $X \vdash x$ for all $x \in X$,
- if $X \vdash y$ for all $y \in Y$ and $Y \vdash z$, then $X \vdash z$.

Given a set of formulas Fm in a language $\mathcal{L}$, a logic L for us is a consequence relation $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(Fm) \times Fm$ on the set of formulas Fm which is substitution invariant, i.e.,

- if $\Phi \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Phi] \vdash_{\mathrm{L}} \sigma[\varphi]$ for all substitutions σ in the language $\mathcal{L}$.

The language $\mathcal{L}_{\mathrm{FL}}$ of the logic FL consists of a countable set of propositional variables, binary connectives $\wedge, \vee, \cdot, \backslash, /$ and constants $0, 1$. The binary connectives are respectively called *lattice conjunction, lattice disjunction, fusion, left and right implication (or division)*. Thus from now on Fm denotes the set of formulas in the language $\mathcal{L}_{\mathrm{FL}}$.

In order to define $\vdash_{\mathrm{FL}}$, we have to say what a sequent is and what it means that a sequent is provable from a set of sequents. A *sequent* is an ordered pair $\Gamma \Rightarrow \varphi$, where Γ is a sequence of formulas (could be also empty) and φ a formula or the empty sequence. A sequent is called *initial* if it has a form of one of the following sequents:

$$\alpha \Rightarrow \alpha \qquad\qquad \Rightarrow 1 \qquad\qquad 0 \Rightarrow$$

where $\alpha \in Fm$. Given a set of sequents S and a sequent s, we say that s *is provable from* S in the sequent calculus for FL if one of the following conditions is satisfied:

- $s \in S$,
- s is initial,
- s can be obtained from S and the initial sequents by application of finitely many rules from Figure 1, where symbols Γ, Σ, Π denote sequences of formulas and α, β, φ formulas.

Now we are ready to define $\vdash_{\mathrm{FL}}$. Let $\Phi \cup \{\psi\} \subseteq Fm$. Then $\Phi \vdash_{\mathrm{FL}} \psi$ holds iff the sequent $\Rightarrow \psi$ is provable from the set of sequents $\{\Rightarrow \varphi \mid \varphi \in \Phi\}$. Analogously we say that ψ *is provable from* Φ (also Φ *proves* ψ) in the logic FL if $\Phi \vdash_{\mathrm{FL}} \psi$. We will illustrate the above notions on an example. Let χ and α be formulas. Then Figure 2 shows that $\{\chi\} \vdash_{\mathrm{FL}} (\alpha\backslash(\chi \cdot \alpha)) \wedge 1$.

Let Φ be a set of formulas understood as axiom schemata. An *axiomatic extension* $\mathrm{FL} + \Phi$ of FL by the set of axiom schemata Φ is the logic defined by the same sequent calculus as FL enriched by new initial sequents

$$\{\Rightarrow \sigma(\varphi) \mid \varphi \in \Phi \text{ and } \sigma \text{ is a substitution in the language } \mathcal{L}_{\mathrm{FL}}\}\,.$$

The consequence relation $\vdash_{\mathrm{FL}+\Phi}$ is defined analogously as $\vdash_{\mathrm{FL}}$ using this sequent calculus. Many of the substructural logics (among them also semilinear logics) can be viewed as axiomatic extensions of FL. Thus in this text we will call a logic L *substructural* if L is an axiomatic extension of FL.

Table 1 shows a few basic axiom schemata together with their names. Let $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. We denote by FL_{S} the axiomatic extension of FL by axiom schemata from S, e.g. $\mathrm{FL}_{\mathrm{ci}}$ denotes axiomatic extension of FL by (c) and (i). Note that (w) is an abbreviation for (i) together with (o). Thus we write for instance $\mathrm{FL}_{\mathrm{ew}}$ instead of $\mathrm{FL}_{\mathrm{eio}}$. The logics FL_{S} are called *basic substructural logics*.

REMARK 1.0.1. *We have introduced the basic substructural logics* FL_{S} *as axiomatic extensions of* FL. *In proof theory they are usually equivalently presented as extensions of* FL *by combinations of corresponding structural rules of* exchange *(e),* contraction *(c),* left weakening *(i) and* right weakening *(o), see Figure 3. Thus* $\mathrm{FL}_{\mathrm{ecw}}$ *is nothing else but* intuitionistic logic.

$$\frac{\Gamma \Rightarrow \alpha \qquad \Sigma, \alpha, \Pi \Rightarrow \varphi}{\Sigma, \Gamma, \Pi \Rightarrow \varphi}\ (\text{cut})$$

$$\frac{\Gamma, \Sigma \Rightarrow \varphi}{\Gamma, 1, \Sigma \Rightarrow \varphi}\ (1\Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \varphi}\ (\wedge\Rightarrow) \qquad \frac{\Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \varphi}\ (\wedge\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi \qquad \Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \varphi}\ (\vee\Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta, \Sigma \Rightarrow \varphi}\ (\cdot\Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \varphi}{\Pi, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \varphi}\ (\backslash\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow\backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \varphi}{\Pi, \beta/\alpha, \Gamma, \Sigma \Rightarrow \varphi}\ (/\Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow/)$$

Figure 1. The rules of the sequent calculus for the logic FL.

$$\dfrac{\dfrac{\dfrac{\Rightarrow \chi \qquad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \chi \cdot \alpha}\ (\Rightarrow\cdot)}{\Rightarrow \alpha\backslash(\chi \cdot \alpha)}\ (\Rightarrow\backslash) \qquad \Rightarrow 1}{\Rightarrow \alpha\backslash(\chi \cdot \alpha) \wedge 1}\ (\Rightarrow\wedge)$$

Figure 2. The proof of $(\alpha\backslash(\chi \cdot \alpha)) \wedge 1$ from $\{\chi\}$.

Name	Axiom schema(ta)
(e)	$(\alpha \cdot \beta)\backslash(\beta \cdot \alpha)$
(c)	$\alpha\backslash(\alpha \cdot \alpha)$
(i)	$\alpha\backslash 1$
(o)	$0\backslash\alpha$
(w)	$\alpha\backslash 1, 0\backslash\alpha$

Table 1. Axioms schemata of basic substructural logics.

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \varphi}\ (\mathrm{e}) \qquad \frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \varphi}{\Gamma, \alpha, \Sigma \Rightarrow \varphi}\ (\mathrm{c})$$

$$\frac{\Gamma, \Sigma \Rightarrow \varphi}{\Gamma, \alpha, \Sigma \Rightarrow \varphi}\ (\mathrm{i}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}\ (\mathrm{o})$$

Figure 3. The structural rules of exchange (e), contraction (c), left weakening (i) and right weakening (o).

Substructural logics can be partially ordered by their strength. More precisely, this partial order is just the inclusion ordering, i.e., a logic L_2 is *stronger* than a logic L_1 if $\vdash_{\mathrm{L}_1} \subseteq \vdash_{\mathrm{L}_2}$. In fact, this order is a complete lattice order, i.e., the substructural logics form a complete lattice $\mathbf{\Lambda}(\mathrm{FL})$. The join $\bigvee_{i\in I} \mathrm{L}_i$ of a collection of substructural logics $\{\mathrm{L}_i \mid i \in I\}$, where $\mathrm{L}_i = \mathrm{FL} + \Phi_i$ for some $\Phi_i \subseteq Fm$, is the substructural logic axiomatized by the union of Φ_i's, i.e., $\bigvee_{i\in I} \mathrm{L}_i = \mathrm{FL} + \bigcup_{i\in I} \Phi_i$. A description of meets is not so easy but it can be done as shown in Chapter II. Further, note that FL is the bottom element of $\mathbf{\Lambda}(\mathrm{FL})$. The top element is the inconsistent logic which proves everything. Close to the top element is located *classical logic* which is one of the maximally consistent substructural logics. Figure 4 depicts the ordering from $\mathbf{\Lambda}(\mathrm{FL})$ for basic substructural logics. Note that $\mathrm{FL}_{\mathrm{ci}} = \mathrm{FL}_{\mathrm{eci}}$ and $\mathrm{FL}_{\mathrm{cw}} = \mathrm{FL}_{\mathrm{ecw}}$ since (e) can be proved in the presence of (c) and (i).

As we mentioned at the beginning we are interested in substructural logics which are complete with respect to a class of linearly ordered algebras, so-called *semilinear logics*. Since this is mainly a semantical notion, we postpone a formal definition of semilinear logics until we define algebraic semantics for substructural logics.

2 Algebraic preliminaries

We have to recall several definitions and results. First, we recall basic facts from universal algebra. Second, we recall the notion of a residuated map which is an essential ingredient of our algebraic semantics for substructural logics. Finally, a closely related notion of a Galois connection is discussed. The sets of natural numbers, integers, rational and real numbers are denoted $\mathsf{N}, \mathsf{Z}, \mathsf{Q}, \mathsf{R}$ respectively.

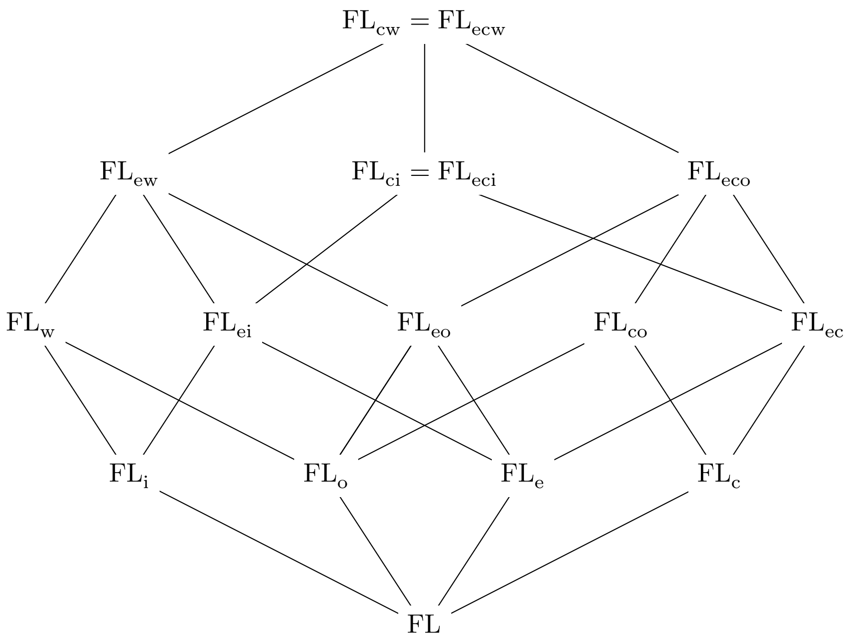

Figure 4. Basic substructural logics.

2.1 Universal algebra

We assume the reader is familiar with the basics of universal algebra, most of which can be found in [6]. Thus we will recall only a necessary minimum to present the results in subsequent sections. Let $\boldsymbol{A}$ be an algebra for an algebraic language $\mathcal{L}$, i.e., $\boldsymbol{A} = \langle A, \langle f^{\boldsymbol{A}} \mid f \in \mathcal{L}\rangle\rangle$ is a set A endowed with a set of operations indexed by the connectives from $\mathcal{L}$ such that corresponding to each n-ary connective f there is an n-ary operation $f^{\boldsymbol{A}}\colon A^n \to A$. Given a term t in the language $\mathcal{L}$, the corresponding term function on $\boldsymbol{A}$ is denoted $t^{\boldsymbol{A}}$ or shortly t if $\boldsymbol{A}$ is clear from the context. We will be interested mainly in the case when $\mathcal{L} = \mathcal{L}_{\mathrm{FL}}$. Thus the set of terms will be typically just the set of formulas Fm in the language $\mathcal{L}_{\mathrm{FL}}$.

Let us fix an algebraic language $\mathcal{L}$. In the rest of this section all algebras will be algebras for the language $\mathcal{L}$. Having an algebra $\boldsymbol{A}$, we can define the *equational consequence relation* $\models_{\boldsymbol{A}}$. Recall that an evaluation into $\boldsymbol{A}$ is a homomorphism from the term algebra (i.e., the absolutely free algebra) into $\boldsymbol{A}$ determined uniquely by the images of variables. Let $E = \{t_i = s_i \mid i \in I\}$ be a set of identities and $t = s$ an identity. Then

$$E \models_{\boldsymbol{A}} t = s \quad \text{iff} \quad \begin{array}{l}\text{for all evaluations } e \text{ we have } e(t) = e(s)\\ \text{whenever } e(t_i) = e(s_i) \text{ for all } i \in I.\end{array}$$

If $E = \emptyset$ then we write only $\models_{\boldsymbol{A}} t = s$ instead of $\emptyset \models_{\boldsymbol{A}} t = s$. The definition of $\models_{\boldsymbol{A}}$ can be extended to an arbitrary class of algebras $\mathbb{K}$, by $E \models_{\mathbb{K}} t = s$ iff $E \models_{\boldsymbol{A}} t = s$

holds for every $\boldsymbol{A} \in \mathbb{K}$.

Having defined the consequence relation in an algebra, we may define an *equational class* of algebras. Given a set of identities $E = \{t_i = s_i \mid i \in I\}$, the equational class defined by E is the class of all algebras $\boldsymbol{A}$ such that $\models_{\boldsymbol{A}} t_i = s_i$ holds for all $i \in I$. Similarly, one can define a *quasi-equational class* of algebras. Given a set of quasi-identities Q, the quasi-equational class defined by Q is the class of all algebras $\boldsymbol{A}$ such that for every quasi-identity

$$t_1 = s_1 \quad \text{and} \quad \cdots \quad \text{and} \quad t_n = s_n \qquad \text{implies} \qquad t = s$$

from Q we have $\{t_i = s_i \mid i = 1, \dots, n\} \models_{\boldsymbol{A}} t = s$.

Equational and quasi-equational classes can be characterized by means of class operators. Let $\mathbb{K}$ be a class of algebras. Then we define $\mathbf{I}(\mathbb{K})$, $\mathbf{H}(\mathbb{K})$, $\mathbf{S}(\mathbb{K})$, $\mathbf{P}(\mathbb{K})$ and $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$ to be, respectively, the class of all algebras isomorphic to some member of $\mathbb{K}$, the class of all homomorphic images of members from $\mathbb{K}$, the class of all subalgebras of members from $\mathbb{K}$, the class of all direct products of members from $\mathbb{K}$ and the class of all ultraproducts of members from $\mathbb{K}$. The class $\mathbb{K}$ is called a *variety* if it is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$. If $\mathbb{K}$ is closed under $\mathbf{I}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_{\mathrm{U}}$ and contains a trivial algebra, then $\mathbb{K}$ is said to be a *quasivariety*. It is known that the smallest variety (resp. quasivariety) containing a class $\mathbb{K}$ is the class $\mathbf{HSP}(\mathbb{K})$ (resp. $\mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$). That is why we shortly denote the compositions of operators $\mathbf{HSP}$ and $\mathbf{ISPP}_{\mathrm{U}}$ respectively $\mathbf{V}$ and $\mathbf{Q}$.

It is very well known due to Birkhoff that varieties are precisely equational classes. An analogous result for quasivarieties was proved by Mal'cev.

THEOREM 2.1.1. *Let $\mathbb{K}$ be a class of algebras. Then the following hold:*

1. *$\mathbb{K}$ is a variety (i.e., $\mathbb{K} = \mathbf{V}(\mathbb{K})$) iff it is an equational class.*
2. *$\mathbb{K}$ is a quasivariety (i.e., $\mathbb{K} = \mathbf{Q}(\mathbb{K})$) iff it is a quasi-equational class.*

Another important concept from universal algebra is that of a subdirect product and a subdirectly irreducible algebra.

DEFINITION 2.1.2. *An algebra $\boldsymbol{A}$ is a* subdirect product *of a family $\langle \boldsymbol{A}_i \mid i \in I \rangle$ of algebras if the following hold:*

1. *$\boldsymbol{A}$ is a subalgebra of $\prod_{i \in I} \boldsymbol{A}_i$,*
2. *$\pi_i(\boldsymbol{A}) = \boldsymbol{A}_i$ for all $i \in I$, where π_i denotes the projection to the i-th component.*

Given an algebra $\boldsymbol{A}$, a family $\langle \boldsymbol{A}_i \mid i \in I \rangle$ of algebras and an embedding $f\colon A \to \prod_{i \in I} A_i$, we say the f is *subdirect* if $f[\boldsymbol{A}]$ is a subdirect product of $\langle \boldsymbol{A}_i \mid i \in I \rangle$.

DEFINITION 2.1.3. *An algebra $\boldsymbol{A}$ is said to be* subdirectly irreducible *if it is nontrivial and for every subdirect embedding $f\colon A \to \prod_{i \in I} A_i$ there is $i \in I$ such that $\pi_i \circ f\colon A \to A_i$ is an isomorphism.*

Subdirectly irreducible algebras can be characterized by means of their congruence lattices. Let $\boldsymbol{A}$ be a nontrivial algebra and $\boldsymbol{Con}(\boldsymbol{A})$ its congruence lattice. Then $\boldsymbol{A}$

is subdirectly irreducible iff the bottom element Δ of $\boldsymbol{Con}(\boldsymbol{A})$ is completely meet-irreducible. Recall that an element a in a lattice is called completely meet-irreducible if $a = \bigwedge_{i\in I} a_i$ implies $a = a_i$ for some $i \in I$.

THEOREM 2.1.4. *A nontrivial algebra $\boldsymbol{A}$ is subdirectly irreducible iff Δ is completely meet irreducible, i.e., $Con(\boldsymbol{A})\backslash\{\Delta\}$ has a minimum.*

The minimum of $Con(\boldsymbol{A})\backslash\{\Delta\}$ from the previous theorem is called the *monolith* of $\boldsymbol{A}$. Subdirectly irreducible algebras are important because they are building blocks of any algebra. Precisely, we have the following theorem.

THEOREM 2.1.5. *Every nontrivial algebra $\boldsymbol{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebras (which are homomorphic images of $\boldsymbol{A}$).*

An important consequence of the previous theorem is that the class of subdirectly irreducible algebras inside a variety defines the same consequence relation as the whole variety. Let $\mathbb{K}$ be a variety. The class of its subdirectly irreducible members is denoted $\mathbb{K}_{\mathrm{SI}}$. We start with a lemma which is easy to prove.

LEMMA 2.1.6. *Let $\mathbb{K}$ be a class of algebras. Then $\models_{\mathbf{O}(\mathbb{K})} \; = \; \models_{\mathbb{K}}$ for any operator $\mathbf{O} \in \{\mathbf{I}, \mathbf{S}, \mathbf{P}\}$.*

PROPOSITION 2.1.7. *Let $\mathbb{K}$ be a variety. Then $\models_{\mathbb{K}} = \models_{\mathbb{K}_{\mathrm{SI}}}$.*

Proof. Clearly $\models_{\mathbb{K}} \subseteq \models_{\mathbb{K}_{\mathrm{SI}}}$ since $\mathbb{K} \supseteq \mathbb{K}_{\mathrm{SI}}$. Conversely, assume $E \models_{\mathbb{K}_{\mathrm{SI}}} t = s$. We have to show that $E \models_{\mathbb{K}} t = s$. Let $\boldsymbol{A} \in \mathbb{K}$. By Theorem 2.1.5 the algebra $\boldsymbol{A}$ is isomorphic to a subalgebra of $\prod_{i\in I} \boldsymbol{A}_i$ where $\boldsymbol{A}_i$'s are subdirectly irreducible and homomorphic images of $\boldsymbol{A}$. Since varieties are closed under homomorphic images, we must have $\boldsymbol{A}_i \in \mathbb{K}_{\mathrm{SI}}$ for all $i \in I$. The rest follows by Lemma 2.1.6. $\square$

The next theorem is known as *Jónsson's lemma*. Recall that a variety $\mathbb{L}$ is called *congruence distributive* if for all $\boldsymbol{A} \in \mathbb{L}$ the congruence lattice $\boldsymbol{Con}(\boldsymbol{A})$ is distributive.

THEOREM 2.1.8. *Let $\mathbf{V}(\mathbb{K})$ be a congruence distributive variety generated by a class $\mathbb{K}$ and $\boldsymbol{A} \in \mathbf{V}(\mathbb{K})$. If $\boldsymbol{A}$ is subdirectly irreducible, then $\boldsymbol{A} \in \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$.*

It is well known that varieties whose members have a lattice reduct are congruence distributive. This is important for us since most of our algebras will have a lattice reduct so that the above theorem applies to them.

Finally, we need to recall several definitions on partial subalgebras and partial embeddings.

DEFINITION 2.1.9. *Let $\boldsymbol{A} = \langle A, \langle f_i^{\boldsymbol{A}} \mid i \in I\rangle\rangle$ be an algebra and $\emptyset \neq G \subseteq A$. The* partial subalgebra *$\boldsymbol{G}$ of $\boldsymbol{A}$ is the partial algebra $\boldsymbol{G} = \langle G, \langle f_i^{\boldsymbol{G}} \mid i \in I\rangle\rangle$, where for every n-ary operation f_i and $a_1, \ldots, a_n \in G$ we have*

$$f_i^{\boldsymbol{G}}(a_1, \ldots, a_n) = \begin{cases} f_i^{\boldsymbol{A}}(a_1, \ldots, a_n) & \text{if } f_i^{\boldsymbol{A}}(a_1, \ldots, a_n) \in G, \\ \text{undefined} & \text{if } f_i^{\boldsymbol{A}}(a_1, \ldots, a_n) \notin G. \end{cases}$$

Clearly, every usual subalgebra is also a partial subalgebra whose operations are defined everywhere.

DEFINITION 2.1.10. *Let $\boldsymbol{A} = \langle A, \langle f_i^{\boldsymbol{A}} \mid i \in I\rangle\rangle$, $\boldsymbol{B} = \langle B, \langle f_i^{\boldsymbol{B}} \mid i \in I\rangle\rangle$ be algebras of the same type and $\boldsymbol{G}$ a partial subalgebra of $\boldsymbol{A}$. A one-to-one map $h\colon G \to B$ is called a* partial embedding *if it preserves all existing operations on $\boldsymbol{G}$, i.e., for every n-ary operation f_i and $a_1, \ldots, a_n \in G$ if $f_i^{\boldsymbol{G}}(a_1, \ldots, a_m)$ is defined then*

$$h(f_i^{\boldsymbol{G}}(a_1, \ldots, a_m)) = f_i^{\boldsymbol{B}}(h(a_1), \ldots, h(a_m))\,.$$

Let $\boldsymbol{G}$ be a partial subalgebra of an algebra $\boldsymbol{A}$ and $f\colon G \to B$ a partial embedding from $\boldsymbol{G}$ to an algebra $\boldsymbol{B}$. Observe that the restriction of f to a nonempty subset $G' \subseteq G$ gives a partial embedding from the partial subalgebra $\boldsymbol{G}'$ to $\boldsymbol{B}$. In particular, if $\boldsymbol{G} = \boldsymbol{A}$ (i.e., f is the usual embedding) then the restriction of f to G' gives a partial embedding.

DEFINITION 2.1.11. *Let $\mathbb{K} \cup \{\boldsymbol{A}\}$ be a class of algebras of the same type. We say that $\boldsymbol{A}$ is* partially embeddable *into $\mathbb{K}$ if for every finite partial subalgebra $\boldsymbol{G}$ of $\boldsymbol{A}$ there is a partial embedding $f\colon G \to B$ for some $\boldsymbol{B} \in \mathbb{K}$.*

2.2 Residuated maps and Galois connections

For details on the notions and results from this section see [11, 5, 14]. Before we introduce the notion of a residuated map, we recall several common notions from order theory. Let $\boldsymbol{P} = \langle P, \leq\rangle$ be a partially ordered set (shortly poset). A subset $X \subseteq P$ is called a *downset* if X is closed downwards, i.e., for all $x, y \in P$ we have $x \in X$ and $y \leq x$ implies $y \in X$. Analogously, $Y \subseteq P$ is an *upset* if Y is closed upwards. The sets of all downsets and upsets of $\boldsymbol{P}$ form posets ordered by inclusion. The poset of all downsets (resp. upsets) of $\boldsymbol{P}$ is denoted $\mathcal{D}(\boldsymbol{P})$ (resp. $\mathcal{U}(\boldsymbol{P})$).

Given a subset $S \subseteq P$ we can find the smallest downset ${\downarrow}S$ containing S as the intersection of all downsets containing S, i.e.,

$${\downarrow}S = \bigcap\{X \in \mathcal{D}(\boldsymbol{P}) \mid S \subseteq X\}\,.$$

We can also describe ${\downarrow}S$ by means of principal downsets. By a *principal downset* we mean a downset of the form ${\downarrow}\{x\} = \{y \in P \mid y \leq x\}$ for some $x \in P$. We shortly write ${\downarrow}x$ instead of ${\downarrow}\{x\}$. Then ${\downarrow}S = \bigcup_{x\in S} {\downarrow}x$. Dually we can define the smallest upset ${\uparrow}S$ containing a subset S and a *principal upset* ${\uparrow}x$.

Other well-known useful notions are those of a closure and an interior operator. A map $\gamma\colon P \to P$ is called a *closure operator* on $\boldsymbol{P}$ if it satisfies the following conditions:

- γ is order-preserving, i.e., $x \leq y$ implies $\gamma(x) \leq \gamma(y)$,
- γ is expanding, i.e., $x \leq \gamma(x)$, and
- idempotent, i.e., $\gamma(\gamma(x)) = \gamma(x)$.

Dually, an *interior operator* on $\boldsymbol{P}$ is a map $\sigma\colon P \to P$ which is

- order-preserving,

- contracting, i.e., $\sigma(x) \leq x$, and
- idempotent.

Elements in $\gamma[P]$ are called γ-*closed* or just *closed* if the closure operator is clear from the context. Similarly, elements from $\sigma[P]$ are called σ-*open* or shortly *open*. A subset of closed elements $B \subseteq \gamma[P]$ (resp. open elements $B \subseteq \sigma[P]$) is called a *basis* if every element in $\gamma[P]$ (resp. $\sigma[P]$) can be expressed as a meet (resp. join) of elements from B.

Let δ be a closure or interior operator on $\boldsymbol{P}$. Its image $\delta[P]$ forms a subposet $\delta[\boldsymbol{P}]$ of the poset $\boldsymbol{P}$. Let γ be a closure operator and σ an interior operator on $\boldsymbol{P}$. The closure and interior operators are completely determined by their images $\gamma[P]$ and $\sigma[P]$. Namely, for $x \in P$,

$$\gamma(x) = \min\{a \in \gamma[P] \mid x \leq a\}\,, \qquad \sigma(x) = \max\{a \in \sigma[P] \mid a \leq x\}\,. \tag{1}$$

Conversely any subsets $C, O \subseteq P$ induce respectively a closure operator γ and an interior operator σ if the following minima and maxima exist for every $x \in P$:

$$\gamma(x) = \min\{a \in C \mid x \leq a\}\,, \qquad \sigma(x) = \max\{a \in O \mid a \leq x\}\,. \tag{2}$$

The posets $\gamma[\boldsymbol{P}]$ and $\sigma[\boldsymbol{P}]$ are called respectively *closure* and *interior systems* of $\boldsymbol{P}$.

The following definition of a residuated map will be crucial when we define the algebraic semantics for substructural logics since the operation interpreting the fusion $\cdot$ is in some sense residuated as we will see later.

DEFINITION 2.2.1. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be posets. A map $f\colon P \to Q$ is called* residuated *if there exists a map $f^\dagger\colon Q \to P$ such that for all $p \in P$ and $q \in Q$ we have*

$$f(p) \leq q \text{ iff } p \leq f^\dagger(q).$$

In the above case, we say that f and $f^\dagger$ form a residuated pair. *The map $f^\dagger$ is called a* residual *of f.*

Note that f and $f^\dagger$ form an adjunction between posets $\boldsymbol{P}$ and $\boldsymbol{Q}$ when we view them as categories, i.e., objects are the elements of the poset and morphisms are given by the order relation.

The next proposition lists well-known properties of residuated maps.

PROPOSITION 2.2.2. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be posets and $f\colon P \to Q$ a residuated map with its residual $f^\dagger$. Then the following hold.*

1. *$f, f^\dagger$ are monotone,*
2. *$f^\dagger(q) = \max\{p \in P \mid f(p) \leq q\}$,*
3. *$f(p) = \min\{q \in Q \mid p \leq f^\dagger(q)\}$,*
4. *f preserves arbitrary existing joins and $f^\dagger$ arbitrary existing meets, i.e., for any $X \subseteq P$ and $Y \subseteq Q$ we have $f(\bigvee X) = \bigvee_{x\in X} f(x)$ and $f^\dagger(\bigwedge Y) = \bigwedge_{y\in Y} f^\dagger(y)$ if $\bigvee X$ and $\bigwedge Y$ exist.*

The above proposition shows that residuated maps preserve arbitrary existing joins and their residuals arbitrary existing meets. This can be further strengthened if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are complete lattices as is shown in the next proposition.

PROPOSITION 2.2.3. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be complete lattices. Then a map $f\colon P \to Q$ is residuated iff f preserves arbitrary joins. Dually, a map $f^\dagger\colon Q \to P$ is a residual of a map $f\colon P \to Q$ iff $f^\dagger$ preserves arbitrary meets.*

A notion closely related to residuated maps is that of a Galois connection.

DEFINITION 2.2.4. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be posets. We say that maps ${}^{\rhd}\colon P \to Q$ and ${}^{\lhd}\colon Q \to P$ form a* Galois connection *if for all $p \in P$ and $q \in Q$ we have*

$$q \leq p^{\rhd} \text{ iff } p \leq q^{\lhd}.$$

The maps ${}^{\rhd}$ and ${}^{\lhd}$ are called polarities *of the Galois connection.*

Observe that polarities of a Galois connection between posets $\boldsymbol{P}$ and $\boldsymbol{Q}$ form also a residuated pair between $\boldsymbol{P}$ and $\boldsymbol{Q}^{\partial}$ where $\boldsymbol{Q}^{\partial}$ denotes the dual poset of $\boldsymbol{Q}$. Namely, ${}^{\rhd}$ is a residuated map and ${}^{\lhd}$ is its residual.

The following are well-known properties of Galois connections. Note that we compose the polarities from left to right since they are written as superscripts of arguments.

PROPOSITION 2.2.5. *Let ${}^{\rhd}$, ${}^{\lhd}$ be a Galois connection between posets $\boldsymbol{P}$ and $\boldsymbol{Q}$. Then the following hold:*

1. *The maps ${}^{\rhd}$ and ${}^{\lhd}$ are both order-reversing.*
2. *The compositions ${}^{\rhd\lhd}\colon P \to P$ and ${}^{\lhd\rhd}\colon Q \to Q$ are both closure operators.*
3. *We have ${}^{\rhd\lhd\rhd} = {}^{\rhd}$ and ${}^{\lhd\rhd\lhd} = {}^{\lhd}$.*

Let A, B be sets. We denote their powersets respectively $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Typical examples of Galois connections from the literature are Galois connections between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a binary relation $R \subseteq A \times B$. The polarities in this case are given by "upper bounds" and "lower bounds" with respect to the relation R. We are going to discuss these examples in more details because we will need them later. However, we present them in a more general setting. Namely, we will replace the sets A, B by posets. Clearly, this is more general since each set can be viewed as a discrete poset (i.e., a set ordered by the identity relation). Further, note that $\mathcal{P}(\boldsymbol{A}) = \mathcal{D}(\boldsymbol{A}) = \mathcal{U}(\boldsymbol{A})$ for a discrete poset $\boldsymbol{A}$.

Let $\boldsymbol{A}, \boldsymbol{B}$ be two posets and $R \subseteq A \times B$ a relation satisfying for all $a, x \in A$ and $b, y \in B$ the following implication:

$$x \leq a \quad \text{and} \quad a\,R\,b \quad \text{and} \quad b \leq y \qquad \text{implies} \qquad x\,R\,y\,. \tag{3}$$

We call a binary relation R satisfying (3) a *poset relation*. The name comes from category theory since poset relations are just the usual relations in the category of posets.

One can use R in order to define a Galois connection between $\mathcal{D}(\boldsymbol{A})$ and $\mathcal{U}(\boldsymbol{A})$ given by maps ${}^{\rhd}\colon \mathcal{D}(\boldsymbol{A}) \to \mathcal{U}(\boldsymbol{B})$ and ${}^{\lhd}\colon \mathcal{U}(\boldsymbol{B}) \to \mathcal{D}(\boldsymbol{A})$. The maps ${}^{\rhd}$ and ${}^{\lhd}$ are defined as follows:

$$\begin{aligned} X^{\rhd} &= \{y \in B \mid (\forall x \in X)(x\ R\ y)\}\,, \\ Y^{\lhd} &= \{x \in A \mid (\forall y \in Y)(x\ R\ y)\}\,. \end{aligned}$$

Observe that by (3) the image $X^{\rhd}$ is an upset in $\boldsymbol{B}$ and $Y^{\lhd}$ is a downset in $\boldsymbol{A}$. To see that ${}^{\rhd}, {}^{\lhd}$ define a Galois connection, consider $X \in \mathcal{D}(\boldsymbol{A})$ and $Y \in \mathcal{U}(\boldsymbol{B})$. Then $Y \subseteq X^{\rhd}$ in fact means that every element $y \in Y$ satisfies the defining condition $(\forall x \in X)(x\ R\ y)$. Thus validity of $Y \subseteq X^{\rhd}$ is equivalent to the validity of the following first-order formula:

$$(\forall y \in Y)(\forall x \in X)(x\ R\ y)\,.$$

Since the validity of the above formula does not depend on the order of quantifiers, it is immediate that the validity of this formula is also equivalent to validity of $X \subseteq Y^{\lhd}$. Consequently, we obtain the following lemma.

LEMMA 2.2.6. *The pair of maps* ${}^{\rhd}\colon \mathcal{D}(\boldsymbol{A}) \to \mathcal{U}(\boldsymbol{B})$ *and* ${}^{\lhd}\colon \mathcal{U}(\boldsymbol{B}) \to \mathcal{D}(\boldsymbol{A})$ *defines a Galois connection, i.e.,*

$$Y \subseteq X^{\rhd} \quad \textit{iff} \quad X \subseteq Y^{\lhd}\,.$$

It follows from Proposition 2.2.5 that the composition ${}^{\rhd\lhd}\colon \mathcal{D}(\boldsymbol{A}) \to \mathcal{D}(\boldsymbol{A})$ is a closure operator. We denote this closure operator γ_R where the subscript R refers to the relation R defining ${}^{\rhd}$ and ${}^{\lhd}$. One can easily find a basis for γ_R as is shown in the following lemma.

LEMMA 2.2.7. *The collection* $\{(\uparrow y)^{\lhd} \mid y \in B\}$*, where* $\uparrow y$ *is a principal upset from* $\mathcal{U}(\boldsymbol{B})$*, forms a basis for* γ_R*.*

Proof. Note that for each upset $Y \in \mathcal{U}(\boldsymbol{B})$ the downset $Y^{\lhd}$ is γ_R-closed since $Y^{\lhd\rhd\lhd} = Y^{\lhd}$ by Proposition 2.2.5. In particular, for each $y \in B$ the downset $(\uparrow y)^{\lhd}$ is γ_R-closed. Moreover, we have

$$Y^{\lhd} = \{x \in A \mid (\forall y \in Y)(x\ R\ y)\} = \bigcap_{y \in Y} \{x \in A \mid x\ R\ y\} = \bigcap_{y \in Y} (\uparrow y)^{\lhd}\,.$$

The last equality follows from (3). $\square$

3 FL-algebras

This chapter introduces an algebraic semantics for the logic FL, namely the class of so-called FL-algebras. We will work only with algebras for the language $\mathcal{L}_{\mathrm{FL}}$ (and its fragments) consisting of binary connectives $\cdot, \backslash, /, \wedge, \vee$ and constants $0, 1$. We call $\cdot, \backslash, /$ respectively *multiplication, left and right division*. The absolutely free algebra for this language is the term algebra $\boldsymbol{Fm} = \langle Fm, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$. When writing terms in this language we will assume in the absence of parentheses that $\cdot$ is performed first followed by $\backslash, /$ and finally $\wedge, \vee$. We write a^n as a shortcut for $a \cdot a \cdots a$ (n times), where $a^0 = 1$. We often write ab instead of $a \cdot b$.

We start with definitions of simple structures adding more and more conditions so that we finally obtain the definition of an FL-algebra. An algebra $\boldsymbol{A} = \langle A, \cdot \rangle$ with a binary operation is called a *groupoid*. If the multiplication $\cdot$ is associative then $\boldsymbol{A}$ is said to be a *semigroup*. If there is a partial order $\leq$ on $\boldsymbol{A}$ and the multiplication $\cdot$ is order-preserving in both arguments (i.e., $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $x, y, z \in A$), then $\boldsymbol{A} = \langle A, \cdot, \leq \rangle$ is called a *partially ordered groupoid* if $\langle A, \cdot \rangle$ is a groupoid and a *partially ordered semigroup* if $\langle A, \cdot \rangle$ is a semigroup. Further, if this partial order is in fact a lattice order and the multiplication distributes over finite joins (i.e., $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$ and $(y \vee z) \cdot x = y \cdot x \vee z \cdot x$ hold for all $x, y, z \in A$), then $\boldsymbol{A} = \langle A, \cdot, \leq \rangle$ is called a *lattice ordered groupoid* (resp. *lattice ordered semigroup*).

Let $\boldsymbol{A} = \langle A, \cdot, \leq \rangle$ be a partially ordered semigroup. If there is a neutral element 1 for the multiplication (i.e., $1 \cdot x = x = x \cdot 1$ holds for all $x \in A$), then $\boldsymbol{A} = \langle A, \cdot, 1, \leq \rangle$ is said to be a *partially ordered monoid* (shortly *pomonoid*). Analogously as for semigroups we define a *lattice ordered monoid* (shortly *ℓ-monoid*) as a pomonoid whose partial order is a lattice order and $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$ and $(y \vee z) \cdot x = y \cdot x \vee z \cdot x$.

Next we will illustrate the impact the theory of residuated maps has on the properties of $\boldsymbol{A}$. Let $\boldsymbol{A} = \langle A, \cdot, \leq \rangle$ be a partially ordered groupoid whose operation $\cdot$ is residuated component-wise, i.e., for every $a \in A$ the unary maps $l_a(x) = a \cdot x$ and $r_a(x) = x \cdot a$ are residuated. Using the residuals $l_a^\dagger$ and $r_a^\dagger$ of l_a and r_a, we can define new binary operations $\backslash, /$ on $\boldsymbol{A}$ as follows:

$$x \backslash z = l_x^\dagger(z)\,, \qquad z/y = r_y^\dagger(z)\,.$$

Then the operations $\cdot$, $\backslash$ and $/$ are connected by the following property which we call the *residuation property*:

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z/y\,.$$

The above condition is in fact often used as the defining condition for residuated binary maps. Precisely, a binary map $\cdot \colon A \times A \to A$ is said to be *residuated* if there exist binary operations $\backslash \colon A \times A \to A$, $/ \colon A \times A \to A$ satisfying the residuation property. Thus we have the following definition.

DEFINITION 3.0.8. *A* residuated partially ordered groupoid *is an algebraic structure* $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \leq \rangle$ *such that* $\langle A, \leq \rangle$ *is a poset and the residuation property*

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z/y\,.$$

is satisfied for all $x, y, z \in A$.

Note that in the previous definition we do not claim that $\langle A, \cdot, \leq \rangle$ is a partially ordered groupoid because it is not necessary. It follows immediately from the residuation property. Indeed, let $x, y, z \in A$ and assume that $x \leq y$. Then $yz \leq yz$ implies $y \leq yz/z$ by the residuation property. Thus we have $x \leq yz/z$. Using the residuation property again, we obtain $xz \leq yz$. Similarly one can prove $zx \leq zy$. It is also easy to prove that both divisions are order-preserving in the numerator and order-reversing in the denominator.

Properties of unary residuated maps can be easily transferred also to the binary case. We can apply Proposition 2.2.2 to get the following proposition.

PROPOSITION 3.0.9. *Let $\boldsymbol{A}$ be a residuated groupoid and $a, b, c \in A$. Then the following hold:*

1. $a\backslash c = \max\{b \in A \mid ab \leq c\}$ *and* $c/b = \max\{a \in A \mid ab \leq c\}$.
2. $a \cdot b = \min\{c \in A \mid b \leq a\backslash c\} = \min\{c \in A \mid a \leq c/b\}$.
3. *Multiplication distributes over any existing join, i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq A$, then so does $\bigvee_{x\in X, y\in Y} xy$, and*
$$(\bigvee X) \cdot (\bigvee Y) = \bigvee_{x\in X, y\in Y} xy\,.$$
4. *Divisions preserve all existing meets in the numerator and convert all existing joins in the denominator to meets, i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq A$, then for any $z \in A$ the following equalities hold (in particular the right-hand sides exist):*
$$z\backslash(\bigwedge Y) = \bigwedge_{y\in Y} z\backslash y\,, \qquad (\bigwedge Y)/z = \bigwedge_{y\in Y} y/z\,,$$
$$(\bigvee X)\backslash z = \bigwedge_{x\in X} x\backslash z\,, \qquad z/(\bigvee X) = \bigwedge_{x\in X} z/x\,.$$

Residuated lattice ordered semigroups, residuated pomonoids and *residuated ℓ-monoids* are defined analogously as residuated partially ordered groupoids. Observe that the residuation property implies that the multiplication distributes over any existing join. Thus having a residuated pomonoid $\boldsymbol{A}$ whose partial order is a lattice order, it follows immediately that $\boldsymbol{A}$ is a residuated ℓ-monoid.

Now we are ready to define FL-algebras which form a complete algebraic semantics for the logic FL as we will see later.

DEFINITION 3.0.10. *An algebra $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ is an* FL-algebra *if*

- $\langle A, \cdot, 1\rangle$ *is a monoid,*
- $\langle A, \wedge, \vee\rangle$ *is a lattice,*
- $a \cdot b \leq c$ *iff* $b \leq a\backslash c$ *iff* $a \leq c/b$, *for all* $a, b, c \in A$,
- 0 *is an arbitrary element of* A.

In other words, an FL*-algebra $\boldsymbol{A}$ is a residuated ℓ-monoid endowed with a constant* 0. *If the lattice order is linear then we call $\boldsymbol{A}$ an* FL-chain. *The class of all* FL*-algebras is denoted* $\mathbb{FL}$.

Although the class of FL-algebras is not defined only by identities, it can be done as is shown in the following theorem. This theorem uses, instead of identities, inequalities. However, since every FL-algebra has a lattice reduct, we can formally consider each inequality $x \leq y$ as an identity $x = x \wedge y$ or $y = x \vee y$.

THEOREM 3.0.11 ([4]). *An algebra $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ is an* FL*-algebra iff it satisfies the equations defining monoids, the equations defining lattices and the following identities:*

1. $x \cdot (x\backslash z \wedge y) \leq z$,
2. $(y \wedge z/x) \cdot x \leq z$,
3. $y \leq x\backslash(x \cdot y \vee z)$,
4. $y \leq (z \vee y \cdot x)/x$.

Therefore, $\mathbb{FL}$ *is a variety.*

Moreover, $\mathbb{FL}$ is a congruence distributive variety. This follows from the fact that each FL-algebra has a lattice reduct. Thus we can show the congruence distributivity for $\mathbb{FL}$ by the same majority term as for lattices (see [6]).

For the reader's convenience the following lemma provides a list of basic properties of FL-algebras which follow easily from Definition 3.0.10.

LEMMA 3.0.12. *The following identities hold in any* FL*-algebra.*

1. $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$,
2. $(x/y)/z = x/zy$ *and* $z\backslash(y\backslash x) = yz\backslash x$,
3. $x\backslash(y/z) = (x\backslash y)/z$,
4. $x/1 = x = 1\backslash x$,
5. $1 \leq x\backslash x$ *and* $1 \leq x/x$,
6. $(z/y)(y/x) \leq z/x$ *and* $(x\backslash y)(y\backslash z) \leq x\backslash z$.

An FL-algebra $\boldsymbol{A}$ is *commutative* if the multiplication is commutative, i.e., $xy = yx$ holds in $\boldsymbol{A}$. We refer to commutative FL-algebras as $\mathrm{FL_e}$*-algebras*. Observe that in $\mathrm{FL_e}$-algebras the third condition in Definition 3.0.10 forces the equality of $a\backslash b$ and b/a. In this case we denote them as $a \to b$.

We call an FL-algebra $\boldsymbol{A}$ *integral* if 1 is a top element, i.e., $\boldsymbol{A}$ satisfies the identity $x \leq 1$. Integral FL-algebras are called shortly $\mathrm{FL_i}$*-algebras*. Similarly, $\mathrm{FL_o}$*-algebras* are FL-algebras where 0 is the bottom element, i.e., they satisfy $0 \leq x$. FL-algebras where the identity $x \leq x^2$ holds are called *contractive* or sometimes *square-increasing*. We refer to them as $\mathrm{FL_c}$*-algebras*. The contractivity of an FL-algebra $\boldsymbol{A}$ implies that the multiplication on its negative elements coincides with the meet. An element $a \in A$

is called *negative* (resp. *positive*) if $a \le 1$ (resp. $a \ge 1$). Let $x, y \in A$ be negative elements. Then $xy \le x1 = x$ and similarly $xy \le y$. Thus $xy \le x \wedge y$. Consequently,

$$x \wedge y \le (x \wedge y)^2 \le xy \le x \wedge y\,.$$

This implies that all contractive integral FL-algebras are commutative.

Observe that the notation $\mathrm{FL_e}$-algebras, $\mathrm{FL_i}$-algebras, $\mathrm{FL_o}$-algebras and $\mathrm{FL_c}$-algebras resembles the notation for basic substructural logics. This is not a coincidence because these algebras will serve as equivalent algebraic semantics for the corresponding substructural logics. Hence we use an analogous notation for names of FL-algebras as for the basic substructural logics. More precisely, let $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. Then $\mathrm{FL_S}$-algebras are FL-algebras satisfying the extra properties listed in S (see Table 2).

In FL-algebras the constant 0 allows us to define the following two unary operations $\sim a = a\backslash 0$ and $-a = 0/a$ which are called respectively *left* and *right negation*. When writing terms containing the negations, we assume that the negations are performed first in the absence of parentheses. Note that the following equivalence holds in any FL-algebra $\boldsymbol{A}$:

$$x \le {\sim} y \qquad \text{iff} \qquad y \le -x\,.$$

The equivalence above follows immediately from the fact that the multiplication in FL-algebras is residuated. It says that $\sim, -$ form a Galois connection. Thus $\sim, -$ are both order-reversing mappings. The next lemma lists basic properties of the negations.

LEMMA 3.0.13. *Let $\boldsymbol{A}$ be an* FL*-algebra and $x \in A$. Then the following hold:*

1. ${\sim}1 = 0 = -1$,
2. $x({\sim}x) \le 0$ *and* $(-x)x \le 0$,
3. $x \le -{\sim}x$ *and* $x \le {\sim}{-}x$,
4. ${\sim}{-}{\sim}x = {\sim}x$ *and* $-{\sim}{-}x = -x$,
5. $x\backslash{\sim}y = {\sim}(yx)$ *and* $-y/x = -(xy)$.

An FL-algebra $\boldsymbol{A}$ is said to be *involutive* if the identities $-{\sim}x = x$ and ${\sim}{-}x = x$ hold in $\boldsymbol{A}$. Consequently, ${\sim}0 = {\sim}{-}1 = 1 = -{\sim}1 = -0$. Moreover, $\sim, -$ become order-reversing bijections. Indeed, ${\sim}x = {\sim}y$ implies $x = -{\sim}x = -{\sim}y = y$. Similarly, $-x = -y$ implies $x = y$. Observe also that the identity $x \le 1$ is equivalent to $0 \le x$ in involutive FL-algebras. Indeed, let $\boldsymbol{A}$ be an involutive FL-algebra and $a \in A$. Assume that the identity $x \le 1$ holds in $\boldsymbol{A}$. Then $-a \le 1$. Thus $0 = {\sim}1 \le {\sim}{-}a = a$. Conversely, suppose that $0 \le x$ holds. Then $0 \le -x$. Thus $x = {\sim}{-}x \le {\sim}0 = 1$. Consequently, the classes of involutive $\mathrm{FL_i}$-algebras, involutive $\mathrm{FL_o}$-algebras and involutive $\mathrm{FL_w}$-algebras are the same.

An FL-algebra $\boldsymbol{A}$ is called *cyclic* if ${\sim}x = -x$. This happens in particular when $\boldsymbol{A}$ is commutative because $\backslash = /$ holds in $\mathrm{FL_e}$-algebras. Thus, every $\mathrm{FL_e}$-algebra is cyclic but the converse need not be true.

Another important class of FL-algebras is the class of residuated lattices. An FL-algebra is called a *residuated lattice* (or shortly RL*-algebra*) if it satisfies the identity

Adjective	*Defining equations*	*Subvariety of* $\mathbb{FL}$	*Subvariety of* $\mathbb{RL}$
Commutative (e)	$x \cdot y = y \cdot x$	$\mathbb{FL}_{\mathrm{e}}$	$\mathbb{RL}_{\mathrm{e}}$
Integral (i)	$x \leq 1$	$\mathbb{FL}_{\mathrm{i}}$	$\mathbb{RL}_{\mathrm{i}}$
Contractive (c)	$x \leq x^2$	$\mathbb{FL}_{\mathrm{c}}$	$\mathbb{RL}_{\mathrm{c}}$
Bounded (o)	$0 \leq x$	$\mathbb{FL}_{\mathrm{o}}$	trivial variety
Integral bounded (w)	$0 \leq x \leq 1$	$\mathbb{FL}_{\mathrm{w}}$	trivial variety
Involutive	$-{\sim}x = x = {\sim}{-x}$	$\mathbb{I}\mathrm{n}\mathbb{FL}$	$\mathbb{I}\mathrm{n}\mathbb{RL}$
Cyclic	$-x = {\sim}x$	$\mathbb{C}\mathrm{y}\mathbb{FL}$	$\mathbb{C}\mathrm{y}\mathbb{RL}$

Table 2. Subvarieties of FL-algebras and residuated lattices.

$1 = 0$. Residuated lattices are usually defined as the 0-free reducts of FL-algebras. However, we choose the above definition in order to capture FL-algebras and residuated lattices within one variety of algebras of the same type.

Since RL-algebras form a subvariety of FL-algebras, any definition for FL-algebras applies also to RL-algebras. Thus we can define RL_{S}-algebras for $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ in the same way as FL_{S}-algebras. However, note that the only RL_{o}-algebra is the trivial one. Indeed, 1 has to be a bottom element in any RL_{o}-algebra $\boldsymbol{A}$. Then it follows that $\boldsymbol{A}$ is trivial by the following lemma.

LEMMA 3.0.14. *Let* $\boldsymbol{A}$ *be a nontrivial* FL*-algebra. Then there is a strictly negative element* $a \in A$*, i.e.,* $a < 1$.

Proof. Since $\boldsymbol{A}$ is nontrivial, there is an element $b \in A$ such that $b \neq 1$. If $1 \not\leq b$ then $a = b \wedge 1 < 1$. If $1 < b$ then we take $a = b\backslash 1$. Clearly we have $a \leq 1\backslash 1 = 1$. Moreover, $a < 1$; otherwise $b = b \cdot a = b \cdot (b\backslash 1) \leq 1$. □

All the properties of FL-algebras and RL-algebras we mentioned so far are summarized in Table 2. Note that classes of FL-algebras and RL-algebras are denoted by the blackboard bold letters, e.g., $\mathbb{FL}_{\mathrm{ew}}$ denotes the variety of $\mathrm{FL}_{\mathrm{ew}}$-algebras or $\mathbb{I}\mathrm{n}\mathbb{FL}_{\mathrm{w}}$ stands for the variety of involutive FL_{w}-algebras. Having an FL-algebra $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$, one can change the interpretation of 0 to 1 in order to define a corresponding RL-algebra $\boldsymbol{A}_r = \langle A, \cdot, \backslash, /, \wedge, \vee, 1, 1\rangle$. We will omit the double 1 in the signature above, i.e., we write just $\boldsymbol{A}_r = \langle A, \cdot, \backslash, /, \wedge, \vee, 1\rangle$.

3.1 Examples of FL-algebras

The variety of FL-algebras encompasses a lot of well-known classes of algebras which were defined and also studied independently. Among these classes are for instance Heyting algebras which can be viewed as $\mathrm{FL}_{\mathrm{cw}}$-algebras. Moreover, involutive Heyting algebras are just Boolean algebras so that $\mathbb{I}\mathrm{n}\mathbb{FL}_{\mathrm{cw}}$ is nothing but the variety of Boolean algebras.

Another well-known class of algebras which can be viewed as FL-algebras are lattice ordered groups (shortly ℓ-groups). The signature of an ℓ-group $\boldsymbol{G}$ is usually written as $\langle G, \wedge, \vee, \cdot, ^{-1}, 1\rangle$. However, ℓ-groups are term equivalent to RL-algebras satisfying an extra identity $x(x\backslash 1) = 1$. The term equivalence translates x^{-1} to $x\backslash 1$ and conversely $x\backslash y$ to $x^{-1}y$ and x/y to xy^{-1}. Let $\boldsymbol{G}$ be an ℓ-group viewed as an RL-algebra. Since RL-algebras are FL-algebras satisfying $1 = 0$, we have $\sim x = x\backslash 1 = x^{-1}$ in $\boldsymbol{G}$. Analogously $-x = 1/x = x^{-1}$. Consequently, $\sim x = -x$. Moreover, $-\sim x = \sim - x = (x^{-1})^{-1} = x$. Thus ℓ-groups in fact form a subvariety of cyclic involutive RL-algebras.

There are also examples of FL-algebras coming from fuzzy logic. Recall that a *t-norm* is a binary operation $*$ on the real unit interval $[0, 1]$ which is associative, commutative, monotone and 1 is its neutral element. Moreover, if $a * (\bigvee Y) = \bigvee_{y\in Y} a * y$ for any $\{a\} \cup Y \subseteq [0, 1]$, $*$ is said to be *left-continuous*. In such a case we also have $(\bigvee Y) * a = \bigvee_{y\in Y} y * a$ by commutativity. Assume that $*$ is left-continuous. Then $*$ is residuated component-wise (see Proposition 2.2.3 and the discussion above Definition 3.0.8). Consequently, there is a binary operation $\to_*$ on $[0, 1]$ such that

$$x * y \leq z \quad \text{iff} \quad y \leq x \to_* z\,.$$

Summing up, we have a residuated commutative pomonoid $\langle [0, 1], *, 1, \leq\rangle$ which is linearly ordered with a top element 1 and a bottom element 0. Thus the algebra $\mathbf{[0, 1]}_* = \langle [0, 1], *, \to_*, \wedge, \vee, 0, 1\rangle$ is just an $\mathrm{FL}_{\mathrm{ew}}$-chain. These algebras are also known as standard MTL-algebras. The word 'standard' refers to the fact that the universe of these algebras is the intended set of truth-degrees $[0, 1]$. Later we will see that the semilinear extension of $\mathrm{FL}_{\mathrm{ew}}$ is complete with respect to $\mathrm{FL}_{\mathrm{ew}}$-chains on $[0, 1]$.

The above definitions works also for the case when $*$ is in addition continuous (in the usual way). Then the corresponding $\mathrm{FL}_{\mathrm{ew}}$-chain $\mathbf{[0, 1]}_*$ is in fact a standard BL-algebra (for more details on BL-algebras see Chapter VII).

3.2 Involutive FL-algebras

We need to recall several properties of involutive FL-algebras which will be useful in the sequel. The most important property is that in involutive FL-algebras the divisions are definable by means of multiplication and negations. Also conversely, the multiplication is definable by divisions and negations.

LEMMA 3.2.1. *Let $\boldsymbol{A}$ be an involutive FL-algebra. Then we have*

1. $x \cdot y = -(y\backslash \sim x) = \sim(-y/x)$,
2. $x\backslash y = \sim[(-y)x]$ *and* $y/x = -[x(\sim y)]$.

Proof. Let $x, y, z \in A$. For the first part we have the following chain of equivalences:

$$xy \leq z = -\sim z \text{ iff } xy(\sim z) \leq 0 \text{ iff } y(\sim z) \leq \sim x \text{ iff } \sim z \leq y\backslash \sim x \text{ iff } -(y\backslash \sim x) \leq z.$$

Then we can prove $xy \leq -(y\backslash \sim x)$ by substituting $-(y\backslash \sim x)$ for z and $-(y\backslash \sim x) \leq xy$ by substituting xy for z. The proof of $xy = \sim(-y/x)$ is analogous.

To prove the second part we can employ the following chain of equivalences:

$$z \leq x\backslash y \text{ iff } xz \leq \sim - y \text{ iff } (-y)xz \leq 0 \text{ iff } (-y)x \leq -z \text{ iff } z \leq \sim[(-y)x].$$

Again by substituting respectively $x\backslash y$ and $\sim[(-y)x]$ for z, we obtain $x\backslash y = \sim[(-y)x]$. The proof of the other equality is analogous. $\square$

Considering the previous lemma, it is natural to ask whether we can define involutive FL-algebras in a restricted language replacing either the division operations or the multiplication by negations $\sim, -$. Both possibilities can be done. However, we will focus only on the second possibility because we will need it later.

A *division lattice* $\boldsymbol{P} = \langle P, \wedge, \vee, \backslash, / \rangle$ is a lattice endowed with two division operations, which satisfy the following condition:

$$y \leq x\backslash z \quad \text{iff} \quad x \leq z/y \,. \tag{4}$$

A division lattice $\boldsymbol{P}$ is said to be *associative* if the identity $x\backslash(y/z) = (x\backslash y)/z$ holds in $\boldsymbol{P}$. Further, we say that $\boldsymbol{P}$ admits a *unit* if there is an element $1 \in P$ such that $1\backslash x = x = x/1$.

An *involutive pair* on a lattice $\boldsymbol{P}$ is a pair of order-reversing operations $\langle \sim, - \rangle$ satisfying the double negation law, i.e., $\sim\!-x = x = -\!\sim\! x$. We call a division lattice *involutive* if it is equipped with an involutive pair $\langle \sim, - \rangle$ and the operations $\sim, -$ are compatible with the divisions, i.e., the following version of contraposition holds:

$$y\backslash -x = \sim\! y/x \,. \tag{5}$$

THEOREM 3.2.2 ([18]). *Involutive* FL*-algebras are term equivalent to associative involutive division lattices that admit a unit.*

The term equivalence in the above theorem is proved via the following translations. Having an involutive FL-algebra, one can define

$$\sim\! x = x\backslash 0 \,, \qquad \text{and} \qquad -x = 0/x \,.$$

Conversely, having an associative involutive division lattice that admits a unit 1, one can define

$$x \cdot y = -(y\backslash \sim\! x) = \sim\!(-y/x) \,, \qquad \text{and} \qquad 0 = \sim\! 1 = -1 \,.$$

3.3 A bottom and a top element

An FL-algebra $\boldsymbol{A}$ need not possess a bottom element, e.g. ℓ-groups are typical examples of FL-algebras without a bottom element. Nevertheless, if $\boldsymbol{A}$ has a bottom element $\bot$ then it has to have also a top element $\top$ since $\bot\backslash\bot = \top$ is a top element of $\boldsymbol{A}$. It is a natural question whether every FL-algebra $\boldsymbol{A}$ can be extended to a lower bounded one. The answer to this question is affirmative.

Let $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1 \rangle$ be an FL-algebra. We describe how to extend $\boldsymbol{A}$ to an FL-algebra $\boldsymbol{A}_\bot$ having a bottom element $\bot$. Let $A_\bot = A \cup \{\bot, \top\}$ with $A \cap \{\bot, \top\} = \emptyset$. The order on $A_\bot$ extends the order on A by setting $\bot \leq a \leq \top$ for every $a \in A$. The multiplication and divisions are extended as follows:

$$\begin{aligned}
\bot \cdot x = x \cdot \bot &= \bot\,, \\
\top \cdot x = x \cdot \top &= \begin{cases} \top & \text{if } x \in A \cup \{\top\}, \\ \bot & \text{if } x = \bot, \end{cases} \\
x \backslash \top = \top / x &= \top\,, \\
\bot \backslash x = x / \bot &= \top\,, \\
\top \backslash x = x / \top &= \begin{cases} \bot & \text{if } x \in A \cup \{\bot\}, \\ \top & \text{if } x = \top, \end{cases} \\
x \backslash \bot = \bot / x &= \begin{cases} \bot & \text{if } x \in A \cup \{\top\}, \\ \top & \text{if } x = \bot. \end{cases}
\end{aligned}$$

Then it can be shown that $\boldsymbol{A}_\bot = \langle A_\bot, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ is an FL-algebra such that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{A}_\bot$. Thus we have the following proposition.

PROPOSITION 3.3.1 ([21]). *Every* FL*-algebra* $\boldsymbol{A}$ *can be embedded in a lower bounded* FL*-algebra* $\boldsymbol{A}_\bot$*, i.e.,* $\boldsymbol{A}_\bot$ *has a bottom and top element. Moreover, the construction of* $\boldsymbol{A}_\bot$ *preserves the following properties of* $\boldsymbol{A}$*: commutativity, contraction and linear order.*

3.4 Algebraizability

In this section we are going to present the most important consequences of the fact that the logic FL is algebraizable. Nevertheless, we will not define the notion of algebraizability precisely (for details see [1] and Chapter II). Roughly speaking, a logic L is *algebraizable* if there is a class of algebras $\mathbb{L}$ (so-called *equivalent algebraic semantics*) satisfying the following conditions:

1. There is a translation τ from formulas in the language of L to identities in the language of members from $\mathbb{L}$. Conversely, there is also a translation ρ from identities to formulas.
2. Let $\Phi \cup \{\varphi\}$ be a set of formulas in the language of L. Then

$$\Phi \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad \tau[\Phi] \models_{\mathbb{L}} \tau(\varphi)\,. \tag{6}$$

3. Let $s = t$ be an identity in the language of $\mathbb{L}$. Then

$$s = t \models_{\mathbb{L}} \tau(\rho(s = t)) \quad \text{and} \quad \tau(\rho(s = t)) \models_{\mathbb{L}} s = t\,. \tag{7}$$

In order to show that the logic FL is algebraizable, one has to first define the translations τ and ρ. Let $\varphi \in Fm$ and $s = t$ an identity in the language $\mathcal{L}_{\mathrm{FL}}$. Then $\tau(\varphi)$ is the identity $1 = 1 \wedge \varphi$ (or shortly $1 \leq \varphi$) and $\rho(s = t)$ is the formula $(s\backslash t) \wedge (t\backslash s)$. Observe that the formulas Fm are just terms in the language $\mathcal{L}_{\mathrm{FL}}$ so the above definition makes sense. Then one has to prove that τ and ρ satisfy the conditions from the definition of an algebraizable logic. This was done in [19]. Further details can be found in Chapter II where it is proved that FL is an algebraizable weakly implicative logic. Such logics are called *algebraically implicative* in Chapter II.

THEOREM 3.4.1. *The translations τ and ρ satisfy (6) and (7) for* $\mathrm{L} = \mathrm{FL}$ *and* $\mathbb{L} = \mathbb{FL}$. *So the logic* FL *is algebraizable and its equivalent algebraic semantics is the variety* $\mathbb{FL}$.

The fact that FL is algebraizable has several important consequences which follow from general results on algebraizable logics. First, there is a dual lattice isomorphism $\mathbb{V}$ between the lattice of substructural logics $\mathbf{\Lambda}(\mathrm{FL})$ and the subvariety lattice $\mathbf{\Lambda}(\mathbb{FL})$ of FL-algebras. Given a substructural logic $\mathrm{L} = \mathrm{FL} + \Phi$, we define $\mathbb{V}(\mathrm{L})$ as the subvariety of FL-algebras satisfying identities from $\tau[\Phi]$. Conversely, given a subvariety $\mathbb{L}$ of FL-algebras axiomatized relatively to $\mathbb{FL}$ by a set of identities E, the inverse $\mathbb{V}^{-1}(\mathbb{L})$ is the axiomatic extension $\mathrm{L} = \mathrm{FL} + \rho[E]$. Second, we have the strong completeness theorem not only for FL but for any axiomatic extension of FL (i.e., any substructural logic).

THEOREM 3.4.2. *Let $\Phi \cup \{\psi\} \subseteq Fm$ and $E \cup \{s = t\}$ a set of identities in the language $\mathcal{L}_{\mathrm{FL}}$. Then the following hold:*

1. *For a substructural logic* L *we have*

$$\Phi \vdash_{\mathrm{L}} \psi \quad \text{iff} \quad \{1 \leq \varphi \mid \varphi \in \Phi\} \models_{\mathbb{V}(\mathrm{L})} 1 \leq \psi\,.$$

2. *For a subvariety* $\mathbb{L}$ *of* FL*-algebras we have*

$$E \models_{\mathbb{L}} s = t \quad \text{iff} \quad \{(u\backslash v) \wedge (v\backslash u) \mid u = v \in E\} \vdash_{\mathbb{V}^{-1}(\mathbb{L})} (s\backslash t) \wedge (t\backslash s)\,.$$

3.5 Congruences and filters

In this section we are going to show that congruences in an FL-algebra $\boldsymbol{A}$ can be characterized by certain subsets of A which are called filters. More precisely, we are going to show that filters form a lattice which is isomorphic to $\boldsymbol{Con}(\boldsymbol{A})$.

Before we define the notion of a filter, we have to introduce conjugates. Let $\boldsymbol{A}$ be an algebra for the language $\mathcal{L}_{\mathrm{FL}}$. Given $a, x \in A$, we define the *left conjugate* and the *right conjugate* of x with respect to a respectively as $\lambda_a(x) = (a\backslash xa) \wedge 1$ and $\rho_a(x) = (ax/a) \wedge 1$. An *iterated conjugate* of x with respect to $a_1, \dots, a_n \in A$ is a composition of the form $\gamma_{a_1}(\gamma_{a_2}(\cdots\gamma_{a_n}(x)\cdots))$, where $\gamma_{a_i} \in \{\lambda_{a_i}, \rho_{a_i}\}$ for every i. Let $X \subseteq A$. Then the set of all iterated conjugates of elements from X is denoted $\Gamma(X)$. If $X = \{x\}$ then $\Gamma(\{x\})$ is abbreviated as $\Gamma(x)$. A subset $X \subseteq A$ is called *normal* if it is closed under all iterated conjugates, i.e., $\Gamma(X) \subseteq X$. Given a subset $X \subseteq A$ we also define its closure under multiplication

$$\Pi(X) = \{x_1 \cdots x_n \in A \mid x_i \in X,\ n \geq 1\} \cup \{1\}\,.$$

The left and right conjugates were defined for any algebra $\boldsymbol{A}$ for the language $\mathcal{L}_{\mathrm{FL}}$. If, in addition, $\boldsymbol{A}$ is an FL-algebra, then $\lambda_1(x) = \rho_1(x) = x \wedge 1$ since $1\backslash(x1) = (1x)/1 = x$ hold in any FL-algebra. Furthermore, $\lambda_a(1) = \rho_a(1) = 1$ because $1 \leq a\backslash a$ and $1 \leq a/a$ (see Lemma 3.0.12).

DEFINITION 3.5.1. *Let $\boldsymbol{A}$ be an* FL*-algebra and $F \subseteq A$. Then F is a* filter *of $\boldsymbol{A}$ if the following hold:*

(u) $1 \in F$

(up) *F is an upset, i.e., $F = \uparrow F$,*

(p) *F is closed under multiplication, i.e., $\Pi(F) \subseteq F$,*

(n) *F is normal, i.e., $\Gamma(F) \subseteq F$.*

It is easy to see that (n) can be simplified when $\boldsymbol{A}$ is commutative. Namely, the condition $\Gamma(F) \subseteq F$ can be replaced by $F \wedge 1 \subseteq F$, where $F \wedge 1 = \{x \wedge 1 \mid x \in F\}$. Indeed, in the commutative case we have

$$(a \backslash xa) \wedge 1 = (a \backslash ax) \wedge 1 \geq x \wedge 1 \,.$$

If $\boldsymbol{A}$ is in addition integral (i.e., an $\mathrm{FL_{ei}}$-algebra), then (n) is redundant.

The notion of a filter was not discovered only by investigating congruences of FL-algebras but it comes from the logic itself. Namely filters of an FL-algebra $\boldsymbol{A}$ are just subsets closed under all deductions; a subset $F \subseteq A$ is closed under all deductions if for all $\Phi \cup \{\varphi\} \subseteq Fm$ and all evaluations $e\colon Fm \to A$ we have the following implication:

$$\Phi \vdash_{\mathrm{FL}} \varphi \quad \text{and} \quad e[\Phi] \subseteq F \qquad \text{implies} \qquad e(\varphi) \in F.$$

For further details on this see the notion of a *logical filter* in Chapter II.

Let $\boldsymbol{A}$ be an FL-algebra. Observe that the collection $Fi(\boldsymbol{A})$ of all filters of $\boldsymbol{A}$ form a poset $\boldsymbol{Fi}(\boldsymbol{A}) = \langle Fi(\boldsymbol{A}), \subseteq \rangle$ ordered by inclusion. It is in fact a complete lattice because it is easy to see that filters are closed under arbitrary intersections. Since $\boldsymbol{Fi}(\boldsymbol{A})$ is a complete lattice, it makes sense to define a least filter $F(X)$ containing a subset $X \subseteq A$ as the intersection of all filters F such that $X \subseteq F$. We also say that $F(X)$ is the filter *generated* by X. It is a natural question whether we can describe elements of $F(X)$ by means of elements from X. The answer is the content of the following theorem.

THEOREM 3.5.2. *Let $\boldsymbol{A}$ be an* FL*-algebra and $X \subseteq A$. Then $F(X) = \uparrow \Pi\Gamma(X)$. In other words, $y \in F(X)$ iff there are elements $x_1, \ldots, x_n \in X$ and iterated conjugates $\gamma_1, \ldots, \gamma_n$ such that $\gamma_1(x_1) \cdots \gamma_n(x_n) \leq y$. If $\boldsymbol{A}$ is commutative then $F(X) = \uparrow \Pi(X \wedge 1)$.*

Now we are about to define the isomorphism between $\boldsymbol{Fi}(\boldsymbol{A})$ and $\boldsymbol{Con}(\boldsymbol{A})$ for an FL-algebra $\boldsymbol{A}$. Let $\theta \in Con(\boldsymbol{A})$ and $F \in Fi(\boldsymbol{A})$. We define two maps $F_c\colon Con(\boldsymbol{A}) \to Fi(\boldsymbol{A})$ and $\Theta_f\colon Fi(\boldsymbol{A}) \to Con(\boldsymbol{A})$ as follows:

$$\begin{aligned} F_c(\theta) &= \uparrow(1/\theta)\,, \\ \Theta_f(F) &= \{\langle a, b\rangle \mid a\backslash b, b\backslash a \in F\}\,. \end{aligned}$$

Note that $\Theta_f(F)$ is the *Leibniz congruence* of the matrix $\langle \boldsymbol{A}, F\rangle$ (see Chapter II). Now we are ready for the promised theorem on filters and congruences.

THEOREM 3.5.3. *Let $\boldsymbol{A}$ be an* FL*-algebra. Then the lattice $\boldsymbol{Fi}(\boldsymbol{A})$ is isomorphic to the congruence lattice $\boldsymbol{Con}(\boldsymbol{A})$ via the mutually inverse maps Θ_f and F_c.*

Let $\boldsymbol{A}$ be an FL-algebra and $F \in Fi(\boldsymbol{A})$. The above theorem justifies the convention of writing $\boldsymbol{A}/F$ instead of $\boldsymbol{A}/\Theta_f(F)$. Analogously, a congruence class of $a \in A$ with respect to $\Theta_f(F)$ is denoted a/F.

3.6 Semilinear varieties of FL-algebras

Now we have all the necessary background to introduce semilinear substructural logics. The notion of a semilinear logic is semantical since it is defined by a property of the corresponding algebraic semantics. Roughly, a semilinear logic should be a logic which is sound and complete with respect to a class of linearly ordered algebras. We will make this definition precise for substructural logics. By algebraization we know that every substructural logic L has an equivalent algebraic semantics $\mathbb{V}(\mathrm{L})$. Moreover, the consequence relation $\vdash_{\mathrm{L}}$ can be equivalently translated to the consequence relation $\models_{\mathbb{V}(\mathrm{L})}$, see Theorem 3.4.2. In particular, L is sound and complete with respect to the class of algebras $\mathbb{V}(\mathrm{L})$. Inside the variety $\mathbb{V}(\mathrm{L})$ we can identify linearly ordered algebras and use them in the definition of a semilinear substructural logic. More precisely, for a variety of FL-algebras $\mathbb{L}$, let us denote the class of all FL-chains from $\mathbb{L}$ by $\mathbb{L}_{\mathrm{C}}$.

DEFINITION 3.6.1. *Let* L *be a substructural logic and* $\mathbb{L}$ *its equivalent algebraic semantics. Then* L *is* semilinear *if* $\models_{\mathbb{L}} \; = \; \models_{\mathbb{L}_{\mathrm{C}}}$. *In that case,* $\mathbb{L}$ *is called a* semilinear variety.

Concerning the above definition, we also refer to Chapter II where semilinear logics are defined as weakly implicative logics complete with respect to the class of its linear models. Thus semilinear logics in Chapter II are understood in a broader context than here because every substructural logic is weakly implicative but not vice versa.

It follows immediately from the definition that semilinear varieties are generated by their chains since we have $\models_{\mathbb{L}} s = t$ iff $\models_{\mathbb{L}_{\mathrm{C}}} s = t$ for any identity $s = t$. One can prove also the converse implication saying that varieties generated by chains are semilinear. To see this, let $\mathbb{C}$ be a class of FL-chains and $\mathbb{K} = \mathbf{V}(\mathbb{C})$. By Theorem 2.1.8 $\mathbb{K}_{\mathrm{SI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{C})$. Furthermore, ultraproducts, subalgebras and homomorphic images of chains are again chains. Thus we have

$$\mathbb{K}_{\mathrm{SI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{C}) \subseteq \mathbb{K}_{\mathrm{C}}\,,$$

showing that subdirectly irreducible members of $\mathbb{K}$ have to be chains. Then using Proposition 2.1.7, we obtain

$$\models_{\mathbb{K}} \subseteq \models_{\mathbb{K}_{\mathrm{C}}} \subseteq \models_{\mathbb{K}_{\mathrm{SI}}} = \models_{\mathbb{K}}.$$

Hence $\models_{\mathbb{K}} \; = \; \models_{\mathbb{K}_{\mathrm{C}}}$, i.e., $\mathbb{K}$ is semilinear. Summing up the above ideas, we obtain the following proposition.

PROPOSITION 3.6.2. *Let* $\mathbb{L}$ *be a variety of* FL*-algebras. Then the following are equivalent:*

- $\mathbb{L}$ *is semilinear,*
- $\mathbb{L} = \mathbf{V}(\mathbb{L}_{\mathrm{C}})$,
- $\mathbb{L}_{\mathrm{SI}} \subseteq \mathbb{L}_{\mathrm{C}}$.

Given a variety $\mathbb{L}$ of FL-algebras, we can find the least variety containing $\mathbb{L}_{\mathrm{C}}$. We denote this variety $\mathbb{L}^{\ell}$. The class $\mathbb{L}^{\ell}$ is clearly the variety generated by $\mathbb{L}_{\mathrm{C}}$, i.e., $\mathbb{L}^{\ell} =$

$\mathbf{V}(\mathbb{L}_\mathrm{C})$. Thus $\mathbb{L}^\ell$ is a semilinear variety by Proposition 3.6.2. Moreover, $\mathbb{L}^\ell$ is the greatest semilinear variety below $\mathbb{L}$. Indeed, assume that $\mathbb{L}^\ell \subseteq \mathbb{K} \subseteq \mathbb{L}$ for a semilinear variety $\mathbb{K}$. Since $\mathbb{K}$ is generated by chains which belong to $\mathbb{L}$, we must have $\mathbb{K}_\mathrm{C} \subseteq \mathbb{L}_\mathrm{C}$. Hence $\mathbb{K} \subseteq \mathbb{L}^\ell$. Let L and L^ℓ be the substructural logics corresponding respectively to the varieties $\mathbb{L}$ and $\mathbb{L}^\ell$ via the dual lattice isomorphism between substructural logics and subvarieties of FL-algebras. Then it follows from the discussion above that L^ℓ is the weakest semilinear substructural logic above L. A reader may want to consult Chapter II where the logic L^ℓ is also discussed. Note that $\mathbb{L}_\mathrm{C}$ is denoted $\mathbf{MOD}^\ell(\mathrm{L})$ in Chapter II.

It is a natural question how to axiomatize the variety $\mathbb{L}^\ell$ (resp. logic L^ℓ). In fact, it is sufficient to axiomatize $\mathbb{FL}^\ell$ because $\mathbb{L}^\ell = \mathbb{FL}^\ell \cap \mathbb{L}$. In order to axiomatize $\mathbb{FL}^\ell$, it suffices by Propositions 2.1.7 and 3.6.2 to find identities expressing the fact that subdirectly irreducible members in the variety given by these identities are linearly ordered.

We start with a useful lemma characterizing intersections of principal filters (i.e., filters generated by singletons). Let $\boldsymbol{A}$ be any algebra for the language $\mathcal{L}_\mathrm{FL}$ and $X, Y \subseteq A$. Then we define $X \vee Y = \{x \vee y \mid x \in X, y \in Y\}$. If $\boldsymbol{A} = \boldsymbol{Fm}$ then $1 = X$ denotes the set of identities $\{1 = \tau \mid \tau \in X\}$. Recall also that $\Gamma(X)$ denotes the set of all iterated conjugates of elements from X.

LEMMA 3.6.3. *Let $\boldsymbol{A}$ be an* FL*-algebra and $x, y \in A$. Then*

$$F(x) \cap F(y) = F(\Gamma(x) \vee \Gamma(y))\,.$$

Proof. ($\supseteq$): Since $F(\Gamma(x)\vee\Gamma(y))$ is the least filter containing $\Gamma(x)\vee\Gamma(y)$, it is sufficient to prove that $a \vee b \in F(x) \cap F(y)$ for all $a \in \Gamma(x)$ and $b \in \Gamma(y)$. Clearly, $a \in F(x)$ since filters are closed under conjugations. Thus $a \leq a \vee b \in F(x)$ because $F(x)$ is an upset. Similarly, $a \vee b \in F(y)$.

($\subseteq$): Let $z \in F(x) \cap F(y)$. By Theorem 3.5.2 there are $k, l \in \mathsf{N}$, $a \in \Gamma(x)$ and $b \in \Gamma(y)$ such that $a^k \leq z$ and $b^l \leq z$. Consequently, $a^k \vee b^l \leq z$. Since multiplication distributes over joins, we get

$$(a \vee b)^{k+l} = \bigvee_{f\colon \{1,\dots,k+l\}\to\{a,b\}} f(1)\cdots f(k+l)\,.$$

As every $f(1)\cdots f(k+l)$ contains either at least k-times a or at least l-times b, it follows that $(a \vee b)^{k+l} \leq a^k \vee b^l$ (note that $a, b \leq 1$ because any conjugate is less than or equal to 1). Thus $(a \vee b)^{k+l} \leq z$. This means that z belongs to the filter generated by $a \vee b \in \Gamma(x) \vee \Gamma(y)$. □

REMARK 3.6.4. *Let $p, q \in Fm$ be propositional variables. It is shown in Chapter II that the set of formulas $\Gamma(p) \vee \Gamma(q)$ forms a* p-disjunction*. The above lemma shows that this p-disjunction satisfies a weak form of the* Proof by Cases Property.

Using Lemma 3.6.3, we can prove that a subdirectly irreducible FL-algebra $\boldsymbol{A}$ is linearly ordered iff it satisfies the following set of identities:

$$1 = \Gamma((x \vee y)\backslash y) \vee \Gamma((x \vee y)\backslash x)\,, \tag{8}$$

where $x, y \in Fm$ are variables and $\Gamma((x \vee y)\backslash y), \Gamma((x \vee y)\backslash x)$ are sets of iterated conjugates in the term algebra $\boldsymbol{Fm}$.

To see that the identities (8) gives an axiomatization for $\mathbb{FL}^\ell$, observe first that in any FL-algebra $\boldsymbol{A}$ we have $x \leq y$ iff $x \vee y \leq y$ iff $1 \leq (x \vee y)\backslash y$. Thus an FL-algebra $\boldsymbol{A}$ is linearly ordered iff it satisfies the following universal sentence:

$$(\forall x)(\forall y)(1 \leq (x \vee y)\backslash y \quad \text{or} \quad 1 \leq (x \vee y)\backslash x)\,.$$

Further, observe that in any FL-algebra $\boldsymbol{A}$ we have $1 \leq x$ implies $\gamma(x) = 1$ for any iterated conjugate γ. Indeed, let $a \in A$. Then $1 \leq x$ implies $a \leq xa$. Thus $1 = a\backslash xa \wedge 1 = \lambda_a(x)$. Similarly, $1 = \rho_a(x)$. Thus also $1 = \gamma(x)$ for any iterated conjugate γ. Consequently, every FL-chain $\boldsymbol{A}$ satisfies the universal sentence

$$(\forall x)(\forall y)(1 = \gamma_1((x \vee y)\backslash y) \quad \text{or} \quad 1 = \gamma_2((x \vee y)\backslash x))\,,$$

for arbitrary iterated conjugates γ_1, γ_2. Then it is easy to see that every FL-chain $\boldsymbol{A}$ has to satisfy all the identities (8), i.e., the identities:

$$\{1 = \gamma_1((x \vee y)\backslash y) \vee \gamma_2((x \vee y)\backslash x) \mid \gamma_1, \gamma_2 \text{ iterated conjugates}\}\,.$$

Conversely, assume that $\boldsymbol{A}$ is a subdirectly irreducible FL-algebra satisfying (8). We want to show that for any $x, y \in A$ either $1 \leq (x \vee y)\backslash y$ or $1 \leq (x \vee y)\backslash x$. By Lemma 3.6.3 we have

$$F((x \vee y)\backslash y) \cap F((x \vee y)\backslash x) = F(\Gamma((x \vee y)\backslash y) \vee \Gamma((x \vee y)\backslash x)) = \uparrow\! 1\,.$$

Since $\boldsymbol{A}$ is subdirectly irreducible, the least filter $\uparrow\! 1$ has to be a completely meet-irreducible element of $\boldsymbol{Fi}(\boldsymbol{A})$ (see Theorems 2.1.4 and 3.5.3). Thus one of the filters $F((x \vee y)\backslash y)$, $F((x \vee y)\backslash x)$ has to be $\uparrow\! 1$. Consequently, either $1 \leq (x \vee y)\backslash y$ or $1 \leq (x \vee y)\backslash x$.

The axiomatization of $\mathbb{FL}^\ell$ given by identities (8) is not finite. Nevertheless, the following theorem shows that we can improve it so that the resulting axiomatization is finite. Note also that Chapter II provides another infinite axiomatization of $\mathbb{FL}^\ell$ using the *prelinearity axioms* similar to (8).

THEOREM 3.6.5. *The class* $\mathbb{FL}^\ell$ *of semilinear* FL*-algebras is axiomatized by the following 4-variable identity:*

$$1 = \lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x)\,. \tag{9}$$

Proof. It is clear that an FL-algebra satisfying $1 = \gamma_1((x \vee y)\backslash y) \vee \gamma_2((x \vee y)\backslash x)$ for all iterated conjugates γ_1, γ_2 satisfies in particular $1 = \lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x)$.

Conversely, let $\boldsymbol{A}$ be an FL-algebra satisfying $1 = \lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x)$. Observe that the following quasi-identities hold in $\boldsymbol{A}$:

$$c \vee d = 1 \quad \Rightarrow \quad \lambda_a(c) \vee d = 1\,, \qquad c \vee d = 1 \quad \Rightarrow \quad c \vee \rho_b(d) = 1\,.$$

Indeed, assume that $c \vee d = 1$. Then $c, d \leq 1$. If we substitute d for x, c for y and 1 for b in (9), we get

$$1 = \lambda_a((d \vee c)\backslash c) \vee \rho_1((d \vee c)\backslash d) = \lambda_a(c) \vee (d \wedge 1) = \lambda_a(c) \vee d\,.$$

Analogously we can derive the second quasi-identity. Finally, applying these quasi-identities on the identity $1 = \lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x)$, we can derive $1 = \gamma_1((x \vee y)\backslash y) \vee \gamma_2((x \vee y)\backslash x)$ for any iterated conjugates γ_1, γ_2. □

The above theorem suggests the following definition.

DEFINITION 3.6.6. *An* FL*-algebra is called* semilinear *if it satisfies the identity*

$$1 = \lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x)\,.$$

The axiomatization of $\mathbb{FL}^\ell$ given in Theorem 3.6.5 can be further simplified if we replace the variety $\mathbb{FL}$ by the variety $\mathbb{FL}_\mathrm{e}$ of commutative FL-algebras. Then we can get rid of the conjugates in (9). Indeed, let $\boldsymbol{A}$ be a semilinear FL_e-algebra. Then $\boldsymbol{A}$ satisfies (9), in particular for $a = b = 1$ we have

$$1 = ((x \vee y)\backslash y \wedge 1) \vee ((x \vee y)\backslash x \wedge 1)\,. \tag{10}$$

Conversely, assume that $\boldsymbol{A}$ is an FL_e-algebra satisfying the identity (10). For all $a, x \in A$ we have $\lambda_a(x) = a\backslash xa \wedge 1 = a\backslash ax \wedge 1 \geq x \wedge 1$. Similarly, we have $\rho_b(y) \geq y \wedge 1$ for $b, y \in A$. Using this observation we can obtain the following chain of inequalities:

$$1 = ((x \vee y)\backslash y \wedge 1) \vee ((x \vee y)\backslash x \wedge 1) \leq \lambda_a((x \vee y)\backslash y) \vee \rho_b((x \vee y)\backslash x) \leq 1\,.$$

Thus the identity (9) holds in $\boldsymbol{A}$, i.e., $\boldsymbol{A}$ is semilinear.

The identity (10) can be simplified a little bit more if one observes that $(x \vee y)\backslash y = x\backslash y \wedge y\backslash y$ and $(x \vee y)\backslash x = x\backslash x \wedge y\backslash x$. Since $1 \leq x\backslash x$ and $1 \leq y\backslash y$, we obtain $(x \vee y)\backslash y \wedge 1 = x\backslash y \wedge 1$ and $(x \vee y)\backslash x = y\backslash x \wedge 1$. Thus the next theorem follows.

THEOREM 3.6.7. *The variety* $\mathbb{FL}_\mathrm{e}^\ell$ *is axiomatized relatively to* $\mathbb{FL}_\mathrm{e}$ *by the identity*

$$1 = (y\backslash x \wedge 1) \vee (x\backslash y \wedge 1)\,. \tag{11}$$

If we want to axiomatize semilinear FL_ei-algebras relatively to $\mathbb{FL}_\mathrm{ei}$ we can even drop the meets with 1 in (11) because 1 is the top element in FL_ei-algebras.

THEOREM 3.6.8. *The variety* $\mathbb{FL}_\mathrm{ei}^\ell$ *is axiomatized relatively to* $\mathbb{FL}_\mathrm{ei}$ *by the identity*

$$1 = y\backslash x \vee x\backslash y\,. \tag{12}$$

As we already mentioned, most fuzzy logics are in fact semilinear substructural logics. However, many of them were introduced independently. Thus they appear in the literature under different names than used here. The same is true also for corresponding algebras of truth values. Table 3 provides a translation table between names of semilinear FL-algebras used here and their original names.

Our name	*Original name*
$\mathrm{FL}^{\ell}_{\mathrm{ew}}$-algebras	MTL-algebras
$\mathrm{FL}^{\ell}_{\mathrm{cw}}$-algebras	Gödel algebras
$\mathrm{FL}^{\ell}_{\mathrm{w}}$-algebras	psMTL^r-algebras
Involutive $\mathrm{FL}^{\ell}_{\mathrm{ew}}$-algebras	IMTL-algebras
Involutive $\mathrm{FL}^{\ell}_{\mathrm{cw}}$-algebras	Boolean algebras
$\mathrm{FL}^{\ell}_{\mathrm{e}}$-algebras (with $\top$,$\bot$)	UL-algebras
Involutive $\mathrm{FL}^{\ell}_{\mathrm{e}}$-algebras (with $\top$,$\bot$)	IUL-algebras
$\mathrm{RL}^{\ell}_{\mathrm{ei}}$-algebras	prelinear semihoops

Table 3. Our names versus original names of semilinear FL-algebras.

3.7 Nuclei and conuclei

Nuclei and conuclei belong to the most useful constructions on FL-algebras (as we will see also in this chapter). They are in fact closure and interior operators on FL-algebras which preserve their semigroup structure in a lax way. A closure operator γ on an FL-algebra $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ is called a *nucleus* if for all $x, y \in A$ we have

$$\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)\,.$$

On the other hand, an interior operator σ on $\boldsymbol{A}$ is said to be a *conucleus*[1] if for all $x, y \in A$ we have

$$\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y) \quad \text{and} \quad \sigma(1) = 1\,.$$

Recall that closure and interior operators are fully determined by their images. The same is true also for nuclei and conuclei.

LEMMA 3.7.1. *Let $\boldsymbol{A}$, $\boldsymbol{B}$ be* FL*-algebras, γ a closure operator on $\boldsymbol{A}$ and σ an interior operator on $\boldsymbol{B}$. Then we have the following:*

1. *γ is a nucleus on $\boldsymbol{A}$ iff $x/y, y\backslash x \in \gamma[A]$ for all $x \in \gamma[A]$, $y \in A$.*
2. *σ is a conucleus on $\boldsymbol{B}$ iff $\sigma[B]$ forms a submonoid of $\boldsymbol{B}$.*

Proof. Suppose that γ is a nucleus. Let $x \in \gamma[A]$ and $y \in A$. We have to show that x/y is γ-closed, i.e., $\gamma(x/y) = x/y$. Clearly $x/y \leq \gamma(x/y)$ because γ is a closure operator. To see the other inequality $\gamma(x/y) \leq x/y$, note that

$$\gamma(x/y)y \leq \gamma(x/y)\gamma(y) \leq \gamma((x/y)y) \leq \gamma(x) = x\,.$$

Thus $\gamma(x/y) \leq x/y$ by the residuation property. Similarly one can prove that $y\backslash x$ is γ-closed. Conversely, suppose that $x/y, y\backslash x \in \gamma[A]$ for all $x \in \gamma[A]$, $y \in A$. We

[1] In the literature there is also a slightly more general definition of a conucleus where the condition $\sigma(1) = 1$ is replaced by a weaker condition $\sigma(1)\sigma(x) = \sigma(x) = \sigma(x)\sigma(1)$.

have to show that $\gamma(x)\gamma(y) \leq \gamma(xy)$. We have $xy \leq \gamma(xy)$ since γ is a closure operator. Thus $x \leq \gamma(xy)/y$. Using the assumption on $\gamma(xy)/y$, we get $\gamma(x) \leq \gamma(xy)/y$. Consequently, $y \leq \gamma(x)\backslash\gamma(xy)$. Again using the assumption, we obtain $\gamma(y) \leq \gamma(x)\backslash\gamma(xy)$. Thus $\gamma(x)\gamma(y) \leq \gamma(xy)$.

To see the second part of the lemma, note that the inequality $\sigma(x)\sigma(y) \leq \sigma(xy)$ is equivalent to the equation $\sigma(\sigma(x)\sigma(y)) = \sigma(x)\sigma(y)$, which says that $\sigma[B]$ is closed under multiplication. Indeed, suppose that $\sigma(x)\sigma(y) \leq \sigma(xy)$ holds. Since σ is idempotent and contracting, we have

$$\sigma(x)\sigma(y) = \sigma(\sigma(x))\sigma(\sigma(y)) \leq \sigma(\sigma(x)\sigma(y)) \leq \sigma(x)\sigma(y)\,.$$

Conversely, assume that $\sigma(\sigma(x)\sigma(y)) = \sigma(x)\sigma(y)$ holds. Then $\sigma(x) \leq x$ and $\sigma(y) \leq y$ implies $\sigma(x)\sigma(y) \leq xy$. Thus we have

$$\sigma(x)\sigma(y) = \sigma(\sigma(x)\sigma(y)) \leq \sigma(xy)\,.$$

Consequently, σ satisfies $\sigma(x)\sigma(y) \leq \sigma(xy)$ iff $\sigma[B]$ is a subsemigroup. Thus the claim follows because $1 \in \sigma[B]$. □

The characterization of nuclei by means of their images given in Lemma 3.7.1(1) can be further improved in the sense that it is sufficient to check the condition only for some elements. Recall that a subset D of a lattice $\boldsymbol{A}$ is called *join-dense* if every element $a \in A$ is equal to a (possibly infinite) join of elements from D.

LEMMA 3.7.2. *Let $\boldsymbol{A}$ be an* FL*-algebra and γ a closure operator on $\boldsymbol{A}$ with a basis B. Further assume that $D \subseteq A$ is a join-dense set in $\boldsymbol{A}$. Then γ is a nucleus iff the following condition holds:*

$$b/d \text{ and } d\backslash b \text{ are } \gamma\text{-closed for all } d \in D \text{ and } b \in B. \tag{13}$$

Proof. By Lemma 3.7.1 the closure operator γ is a nucleus iff $x/y, y\backslash x$ are γ-closed for all $x \in \gamma[A]$, $y \in A$. Clearly, this condition implies (13) since $D \subseteq A$ and $B \subseteq \gamma[A]$. Conversely, assume that (13) holds and consider $x \in \gamma[A]$ and $y \in A$. Since B is a basis of γ we have $x = \bigwedge X$ for some $X \subseteq B$. Similarly, $y = \bigvee Y$ for some $Y \subseteq D$ because D is join-dense. Consequently, we have $y\backslash x = \bigvee Y \backslash \bigwedge X = \bigwedge_{d\in Y} \bigwedge_{b\in X} d\backslash b$ by Proposition 3.0.9. By our assumption $d\backslash b$'s are γ-closed, hence $y\backslash x$ is γ-closed as well because an arbitrary meet of γ-closed elements is again a γ-closed element. Analogously one can show that x/y is γ-closed. □

Let $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ be an FL-algebra and γ a nucleus on $\boldsymbol{A}$. Then the algebra

$$\gamma[\boldsymbol{A}] = \langle \gamma[A], \circ_\gamma, \backslash, /, \wedge, \vee_\gamma, \gamma(0), \gamma(1)\rangle\,,$$

where the operations $\circ_\gamma$ and $\vee_\gamma$ are defined by

$$\begin{aligned} x \circ_\gamma y &= \gamma(x \cdot y)\,, \\ x \vee_\gamma y &= \gamma(x \vee y)\,, \end{aligned}$$

is called the γ-*retraction* (or a *nuclear retraction*) of $\boldsymbol{A}$.

Dually, if σ is a conucleus on $\boldsymbol{A}$ then the algebra

$$\sigma[\boldsymbol{A}] = \langle \sigma[A], \cdot, \backslash_\sigma, /_\sigma, \wedge_\sigma, \vee, \sigma(0), 1 \rangle ,$$

where the operations $\backslash_\sigma$, $/_\sigma$ and $\wedge_\sigma$ are defined by

$$\begin{aligned} x \backslash_\sigma y &= \sigma(x \backslash y) , \\ x /_\sigma y &= \sigma(x/y) , \\ x \wedge_\sigma y &= \sigma(x \wedge y) , \end{aligned}$$

is called the σ-*contraction* (or a *conuclear contraction*) of $\boldsymbol{A}$.

THEOREM 3.7.3 ([18]). *Let $\boldsymbol{A}, \boldsymbol{B}$ be* FL*-algebras, γ a nucleus on $\boldsymbol{A}$, and σ a conucleus on $\boldsymbol{B}$. Then the γ-retraction $\gamma[\boldsymbol{A}]$ and the σ-contraction $\sigma[\boldsymbol{B}]$ are* FL*-algebras.*

The notions of γ-retraction and σ-contraction for FL-chains can be simplified because any subset of a chain is closed under finite meets and joins. Namely, we have $\vee_\gamma = \vee$ and $\wedge_\sigma = \wedge$. Indeed, let $\boldsymbol{A}$ be an FL-chain, γ a nucleus on $\boldsymbol{A}$ and $x, y \in \gamma[A]$. Then either $x \leq y$ or $y \leq x$. Without any loss of generality assume $x \leq y$. Then $\gamma(x \vee y) = \gamma(y) = y = x \vee y$. The equality $\wedge_\sigma = \wedge$ can be proved analogously.

To see how the above notions works, we will present examples of a conucleus and a nucleus. Let $\boldsymbol{A}$ be an FL-algebra and $\sigma\colon A \to A$ a map defined by $\sigma(x) = x \wedge 1$. It is easy to see that σ is an interior operator. Moreover, σ is a conucleus since $\sigma[A]$ forms a submonoid of $\boldsymbol{A}$. Indeed, if $x, y \in \sigma[A]$ then $x, y \leq 1$. Thus $xy \leq 1$, i.e., $\sigma(xy) = xy$. Consequently, $\sigma[\boldsymbol{A}]$ is an FL-algebra. It is even an $\mathrm{FL_i}$-algebra as 1 is clearly a top element of $\sigma[\boldsymbol{A}]$. The algebra $\sigma[\boldsymbol{A}]$ is often called the *negative cone* of $\boldsymbol{A}$ and denoted $\boldsymbol{A}^-$.

Now, let $\boldsymbol{B}$ be an $\mathrm{FL_i}$-algebra and $b \in B$. Then the map $\gamma\colon B \to B$ defined by $\gamma(x) = x \vee b$ is clearly a closure operator. Moreover, γ is a nucleus since

$$\gamma(x)\gamma(y) = (x \vee b)(y \vee b) = xy \vee xb \vee by \vee b^2 \leq xy \vee b = \gamma(xy) .$$

The last inequality holds since $\boldsymbol{B}$ is integral thus xb, by, b^2 are all less than or equal to b. Consequently, $\gamma[\boldsymbol{B}]$ is an FL-algebra which clearly remains integral.

One can also combine the above mentioned conucleus σ with the nucleus γ. Assume that $\boldsymbol{B} = \sigma[\boldsymbol{A}]$. Then the combination gives an $\mathrm{FL_i}$-algebra $\gamma[\sigma[\boldsymbol{A}]]$ whose universe is just the interval $[b, 1]$ in the lattice reduct of $\boldsymbol{A}$.

We have seen that nuclei and conuclei can be used in order to construct new FL-algebras. It is also important to know which properties are preserved by these constructions. Let $\boldsymbol{A}$ be an FL-algebra. We present two ways how to show that some property is preserved by nuclei (resp. conuclei).

LEMMA 3.7.4. *Nuclei are homomorphisms with respect to the language $\{\cdot, \vee, 0, 1\}$ and images of nuclei are subalgebras with respect to the language $\{\backslash, /, \wedge\}$. On the other hand, conuclei are homomorphisms with respect to the language $\{\wedge, 0, 1\}$ and subalgebras with respect to the language $\{\cdot, \vee, 1\}$.*

Proof. The claims on subalgebras follow immediately from the definition of a nuclear retraction and a conuclear contraction.

Let $\boldsymbol{A}$ be an FL-algebra and γ a nucleus on $\boldsymbol{A}$. We claim that $\gamma\colon A \to \gamma[A]$ is a $\{\cdot, \vee, 0, 1\}$-homomorphism. The map γ preserves 0 and 1 because $\gamma(0)$ and $\gamma(1)$ are the corresponding constants in $\gamma[\boldsymbol{A}]$. Let $x, y \in A$. We will show that $\gamma(xy) = \gamma(x) \circ_\gamma \gamma(y) = \gamma(\gamma(x)\gamma(y))$. Obviously $\gamma(xy) \leq \gamma(\gamma(x)\gamma(y))$ because γ is expanding and monotone. Conversely, we have $\gamma(x)\gamma(y) \leq \gamma(xy)$. Thus $\gamma(\gamma(x)\gamma(y)) \leq \gamma(\gamma(xy)) = \gamma(xy)$. Further, we will show that $\gamma(x \vee y) = \gamma(x) \vee_\gamma \gamma(y) = \gamma(\gamma(x) \vee \gamma(y))$. Again the inequality $\gamma(x \vee y) \leq \gamma(\gamma(x) \vee \gamma(y))$ is obvious since γ is expanding and monotone. Conversely, we have $\gamma(x), \gamma(y) \leq \gamma(x \vee y)$. Thus $\gamma(x) \vee \gamma(y) \leq \gamma(x \vee y)$. Consequently, $\gamma(\gamma(x) \vee \gamma(y)) \leq \gamma(\gamma(x \vee y)) = \gamma(x \vee y)$.

Finally, let $\boldsymbol{A}$ be an FL-algebra and σ a conucleus on $\boldsymbol{A}$. We will show that σ is a $\{\wedge, 0, 1\}$-homomorphism. The preservation of $0, 1$ is obvious. Let $x, y \in A$. It remains to check that $\sigma(x \wedge y) = \sigma(x) \wedge_\sigma \sigma(y) = \sigma(\sigma(x) \wedge \sigma(y))$. Since σ is contracting and monotone, we have $\sigma(\sigma(x) \wedge \sigma(y)) \leq \sigma(x \wedge y)$. Conversely, $\sigma(x \wedge y) \leq \sigma(x), \sigma(y)$. Thus $\sigma(x \wedge y) \leq \sigma(x) \wedge \sigma(y)$. Consequently, $\sigma(x \wedge y) = \sigma(\sigma(x \wedge y)) \leq \sigma(\sigma(x) \wedge \sigma(y))$. □

Since homomorphic images preserve identities and substructures preserve universal sentences, it follows from the previous lemma that nuclei preserve all identities in the language $\{\cdot, \vee, 0, 1\}$ and all universal sentences in the language $\{\backslash, /, \wedge\}$. On the other hand, conuclei preserve all identities in the language $\{\wedge, 0, 1\}$ and universal sentences in the language $\{\cdot, \vee, 1\}$.

Although nuclei and conuclei are not homomorphisms of FL-algebras, they can be at least partial homomorphisms in some cases. The following two lemmas present such cases which we will need in the sequel. Let $\boldsymbol{A}$ be an FL-algebra. The first lemma shows that a conucleus σ on $\boldsymbol{A}$ is always a partial embedding of the partial subalgebra of σ-open elements into the σ-contraction $\sigma[\boldsymbol{A}]$.

LEMMA 3.7.5. *Let $\boldsymbol{A}$ be an* FL*-algebra and σ a conucleus on $\boldsymbol{A}$. Further, let $\boldsymbol{G}$ be a partial subalgebra of $\boldsymbol{A}$ defined on a set of σ-open elements, i.e., $G \subseteq \sigma[A]$. Then σ restricted to G is a partial embedding from $\boldsymbol{G}$ to the σ-contraction $\sigma[\boldsymbol{A}]$.*

Proof. First note that the restriction of σ to G is just the identity map. Thus it is one-to-one. Second, σ preserves $\wedge$, 0, and 1 by Lemma 3.7.4. We show that σ preserves $\cdot$ and $\vee$. Recall that $\cdot$ and $\vee$ are computed in $\sigma[\boldsymbol{A}]$ as in $\boldsymbol{A}$ because $\sigma[\boldsymbol{A}]$ is a subalgebra of $\boldsymbol{A}$ with respect to the language $\{\cdot, \vee, 1\}$. Let $\star \in \{\cdot, \vee\}$ and $a, b, a \star b \in G$. Then $\sigma(a) = a$, $\sigma(b) = b$ and $\sigma(a \star b) = a \star b$. Thus we have

$$\sigma(a \star b) = a \star b = \sigma(a) \star \sigma(b)\,.$$

Finally, we show that σ preserves the divisions. Let $a, b \in G$. Since $\sigma(a) = a$ and $\sigma(b) = b$, we have

$$\sigma(a \backslash b) = \sigma(\sigma(a) \backslash \sigma(b)) = \sigma(a) \backslash_\sigma \sigma(b)\,.$$

Similarly, $\sigma(a/b) = \sigma(a)/_\sigma \sigma(b)$ for all $a, b \in G$. □

Let $\boldsymbol{A}$, $\boldsymbol{B}$ be FL-algebras and γ a nucleus on $\boldsymbol{B}$. The second lemma shows that if we are able to embed $\boldsymbol{A}$ into $\boldsymbol{B}$ via $f\colon A \to B$ in such a way that $f[A] \subseteq \gamma[B]$, then f becomes an embedding of $\boldsymbol{A}$ into the γ-retraction $\gamma[\boldsymbol{B}]$. Nevertheless, the next lemma is a bit more general than this. It is sufficient to assume that f is only a partial embedding of a partial subalgebra of $\boldsymbol{A}$ into $\boldsymbol{B}$. Moreover, we will formulate the lemma in such a way that it works also for restricted languages. This will be helpful later. For instance, we will use the following lemma in order to show that f is an embedding with respect to all operations but $\vee$. Then we can concentrate only on the preservation of $\vee$ which does not follow from this general lemma.

LEMMA 3.7.6. *Let $\boldsymbol{A}$, $\boldsymbol{B}$ be* FL*-algebras and γ a nucleus on $\boldsymbol{B}$. Further, let $\boldsymbol{A}'$ be a reduct of $\boldsymbol{A}$, $\boldsymbol{G}$ a partial subalgebra of $\boldsymbol{A}'$ and $f\colon G \to B$ a partial embedding such that $f[G] \subseteq \gamma[B]$. Then f is a partial embedding from $\boldsymbol{G}$ to the corresponding reduct of $\gamma[\boldsymbol{B}]$.*

Proof. Since γ behaves like the identity map on $\gamma[B]$, the composition $\gamma \circ f$ equals f. Thus we have to show that $\gamma \circ f\colon G \to \gamma[B]$ preserves all the operations from $\boldsymbol{G}$. If the operation is one of $\cdot, \vee, 0, 1$, then $\gamma \circ f$ preserves it because γ is a $\{\cdot, \vee, 0, 1\}$-homomorphism by Lemma 3.7.4. Let $\star \in \{\backslash, /, \wedge\}$ be an operation from $\boldsymbol{G}$ and $a, b \in G$ such that $a \star b \in G$. Recall that $\star$ is computed in $\gamma[\boldsymbol{B}]$ in the same way as in $\boldsymbol{B}$. Thus we have to show that $\gamma(f(a \star b)) = \gamma(f(a)) \star \gamma(f(b))$. Since $f[G] \subseteq \gamma[B]$, this equality transforms to $f(a \star b) = f(a) \star f(b)$ which holds because f preserves $\star$ by our assumption. □

3.8 Dedekind-MacNeille completion

Completions play an important role in substructural logics. Given an FL-algebra $\boldsymbol{A}$, by a *completion* of $\boldsymbol{A}$ we mean a pair $\langle \boldsymbol{B}, f \rangle$ where $\boldsymbol{B}$ is an FL-algebra whose lattice reduct is complete and $f\colon A \to B$ is an embedding. We will be mainly interested in completions of FL-chains since it will be a crucial ingredient when proving the completeness of semilinear logics with respect to the $[0,1]$-valued semantics. There are many completions of a given FL-algebra $\boldsymbol{A}$. Among them the most important is the so-called *Dedekind-MacNeille completion* because it preserves all existing meets and joins, i.e., $f[\bigwedge S] = \bigwedge f[S]$ and $f[\bigvee S] = \bigvee f[S]$ for any $S \subseteq A$. The completions having the latter property are said to be *regular*.

In what follows we will show how to construct the Dedekind-MacNeille completion of an FL-algebra $\boldsymbol{A}$ into which $\boldsymbol{A}$ is embeddable and whose lattice reduct is the well-known Dedekind-MacNeille completion of the lattice reduct of $\boldsymbol{A}$. We will proceed in two steps. First, we will consider the poset $\mathcal{D}(\boldsymbol{A})$ of all downsets of $\boldsymbol{A}$ ordered by inclusion. In fact, $\mathcal{D}(\boldsymbol{A})$ forms a complete lattice. One can even extend the other operations of the FL-algebra $\boldsymbol{A}$ to $\mathcal{D}(\boldsymbol{A})$ so that $\mathcal{D}(\boldsymbol{A})$ becomes an FL-algebra. Then one could expect that the mapping $x \mapsto {\downarrow}x$ is an embedding but this is not the case because this map need not preserve finite joins. Thus we have to continue with a second step which selects from $\mathcal{D}(\boldsymbol{A})$ only some of the downsets by means of a nucleus.

Let $\boldsymbol{A} = \langle A, \cdot, 0, 1, \leq \rangle$ be a *pointed pomonoid*, i.e., $\langle A, \cdot, 1, \leq \rangle$ is a pomonoid and $0 \in A$. One can define an FL-algebra on the set $\mathcal{D}(\boldsymbol{A})$ of all downsets of $\boldsymbol{A}$. The set

$\mathcal{D}(\boldsymbol{A})$ itself forms a complete lattice since downsets are closed under arbitrary intersections and unions. We can extend the monoid operation from $\boldsymbol{A}$ to $\mathcal{D}(\boldsymbol{A})$ as follows:

$$X \circ Y = \{u \in A \mid (\exists x \in X)(\exists y \in Y)(u \leq xy)\}\,.$$

Note that $X \circ Y$ is in fact the downset generated by the complex product

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}\,.$$

To simplify the notation we write $X \cdot y$ instead of $X \cdot \{y\}$ and similarly $x \cdot Y$ instead of $\{x\} \cdot Y$.

We will see that $\circ$ is residuated with the divisions defined for $X, Y, Z \in \mathcal{D}(\boldsymbol{A})$ as follows:

$$X\backslash Z = \{y \in A \mid (\forall x \in X)(\exists z \in Z)(xy \leq z)\} = \{y \in A \mid X \cdot y \subseteq Z\}\,,$$

$$Z/Y = \{x \in A \mid (\forall y \in Y)(\exists z \in Z)(xy \leq z)\} = \{x \in A \mid x \cdot Y \subseteq Z\}\,.$$

It is easy to see from the definition that both $X\backslash Z$ and Z/Y are downsets. The algebra $\mathcal{D}(\boldsymbol{A}) = \langle \mathcal{D}(\boldsymbol{A}), \circ, \backslash, /, \cap, \cup, \downarrow 0, \downarrow 1\rangle$ is called the *downset monoid* of $\boldsymbol{A}$. The next lemma shows that the downset monoid forms an FL-algebra.

LEMMA 3.8.1. *The algebra* $\mathcal{D}(\boldsymbol{A}) = \langle \mathcal{D}(\boldsymbol{A}), \circ, \backslash, /, \cap, \cup, \downarrow 0, \downarrow 1\rangle$ *is an* FL*-algebra.*

Proof. The multiplication $\circ$ is clearly associative since $\cdot$ is. Namely, for $U, V, W \in \mathcal{D}(\boldsymbol{A})$ we have

$$(U \circ V) \circ W = U \circ (V \circ W) = \{z \in A \mid (\exists u \in U)(\exists v \in V)(\exists w \in W)(z \leq u \cdot v \cdot w)\}\,.$$

To see that $\downarrow 1$ is a neutral element for $\circ$, note the following:

$$U \circ \downarrow 1 = \{z \in A \mid (\exists u \in U)(z \leq u)\} = U\,.$$

The last equality holds because U is a downset. Similarly we have $\downarrow 1 \circ U = U$.

Finally, we have to check that $\circ$ is residuated, i.e.,

$$X \circ Y \subseteq Z \text{ iff } Y \subseteq X\backslash Z \text{ iff } X \subseteq Z/Y$$

for all $X, Y, Z \in \mathcal{D}(\boldsymbol{A})$. Since $X \circ Y = \downarrow(X \cdot Y)$ and Z is a downset, it follows that $X \circ Y \subseteq Z$ holds iff $X \cdot Y \subseteq Z$, i.e., the first-order sentence $(\forall x \in X)(\forall y \in Y)(\exists z \in Z)(xy \leq z)$ is valid. Taking into account that the validity of this sentence does not depend on the order of universal quantifiers, this sentence is equivalent to $Y \subseteq X\backslash Z$ and also to $X \subseteq Z/Y$. $\square$

Note that the construction of the downset monoid $\mathcal{D}(\boldsymbol{A})$ preserves some basic properties of $\boldsymbol{A}$. Namely, it is easy to see that $\mathcal{D}(\boldsymbol{A})$ remains commutative if $\boldsymbol{A}$ is. Also, if $\boldsymbol{A}$ is integral (i.e., it satisfies $x \leq 1$), then $\mathcal{D}(\boldsymbol{A})$ is an $\mathrm{FL_i}$-algebra since $\downarrow 1 = A$. If 0 is a bottom element of $\boldsymbol{A}$ then $\downarrow 0 = \{0\}$ is a bottom element of $\mathcal{D}(\boldsymbol{A})$, i.e., $\mathcal{D}(\boldsymbol{A})$ is an $\mathrm{FL_o}$-algebra. The FL-algebra $\mathcal{D}(\boldsymbol{A})$ is contractive if $x \leq x^2$ holds in $\boldsymbol{A}$. Indeed, given any $x \in X \subseteq A$, we have $x \leq x^2$. Thus $x \in \downarrow(X \cdot X)$, i.e., $X \subseteq X \circ X$. Further, $\mathcal{D}(\boldsymbol{A})$ is clearly a chain if $\boldsymbol{A}$ is linearly ordered. Finally, it is obvious that $\mathcal{D}(\boldsymbol{A})$ is an RL-algebra if $1 = 0$ holds in $\boldsymbol{A}$. Summing up, we have the following lemma.

LEMMA 3.8.2. *Let $\boldsymbol{A}$ be a pointed pomonoid. The following properties of $\boldsymbol{A}$ are shared by $\mathcal{D}(\boldsymbol{A})$:*

1. *commutativity $xy = yx$,*
2. *integrality $x \leq 1$,*
3. *contractivity $x \leq x^2$,*
4. $0 \leq x$*,*
5. *linear order,*
6. $1 = 0$*.*

Let $\boldsymbol{A}$ be an FL-algebra. It has a reduct which forms a pointed partially ordered monoid. Hence Lemma 3.8.1 applies to $\boldsymbol{A}$. Since the lattice reduct of $\mathcal{D}(\boldsymbol{A})$ is complete, it is natural ask whether $\mathcal{D}(\boldsymbol{A})$ could be the first component of a completion $\langle \mathcal{D}(\boldsymbol{A}), f\rangle$ of $\boldsymbol{A}$. A natural choice for the embedding f would be $f(x) = {\downarrow}x$. However, this f need not preserve finite joins. Thus we have to modify $\mathcal{D}(\boldsymbol{A})$ so that f becomes also a join-preserving map. Nevertheless, if $\boldsymbol{A}$ is an FL-chain, then f is an embedding and $\langle \mathcal{D}(\boldsymbol{A}), f\rangle$ is a completion of $\boldsymbol{A}$. However, this completion need not be regular because $\mathcal{D}(\boldsymbol{A})$ could contain a downset of $\boldsymbol{A}$ which has a supremum in $\boldsymbol{A}$. Summing up, we have at least the following proposition.

PROPOSITION 3.8.3. *Let $\boldsymbol{A}$ be an* FL*-algebra. Then the map $f\colon A \to \mathcal{D}(\boldsymbol{A})$ defined by $f(x) = {\downarrow}x$ is an embedding of the $\vee$-free reduct of $\boldsymbol{A}$ into the $\vee$-free reduct of $\mathcal{D}(\boldsymbol{A})$. If, in addition, $\boldsymbol{A}$ is linearly ordered then f is a homomorphism of* FL*-algebras.*

Proof. Since ${\downarrow}x = {\downarrow}y$ implies $x = y$, f has to be one-to-one. The map f clearly preserves the constants 0 and 1. Further, f preserves multiplication because ${\downarrow}x \circ {\downarrow}y = {\downarrow}({\downarrow}x \cdot {\downarrow}y) = {\downarrow}(xy)$. Indeed, it suffices to note that ${\downarrow}x \cdot {\downarrow}y$ has a maximum xy. It is straightforward to check that ${\downarrow}(x \wedge y) = {\downarrow}x \cap {\downarrow}y$. Finally, we will show that f preserves the divisions. We have

$${\downarrow}x\backslash{\downarrow}y = \{z \in A \mid {\downarrow}x \cdot z \subseteq {\downarrow}y\} = \{z \in A \mid xz \leq y\} = {\downarrow}(x\backslash y)\,.$$

Similarly one can show ${\downarrow}x/{\downarrow}y = {\downarrow}(x/y)$.

To prove the additional part, let $x, y \in A$. Then $x \leq y$ or $y \leq x$. Without any loss of generality suppose that $x \leq y$. Then ${\downarrow}(x \vee y) = {\downarrow}y = {\downarrow}x \cup {\downarrow}y$. □

The next step in the Dedekind-MacNeille completion of an FL-algebra $\boldsymbol{A}$ is to choose only some of the downsets from $\mathcal{D}(\boldsymbol{A})$ so that the map $f(x) = {\downarrow}x$ becomes join-preserving. This can be done by a suitable nucleus. Let $\boldsymbol{A} = \langle A, \cdot, 0, 1, \leq\rangle$ be a pointed pomonoid and $\boldsymbol{B} = \langle B, \leq\rangle$ a poset. Recall that every poset relation $R \subseteq A \times B$ defines a closure operator $\gamma_R = {}^{\rhd\lhd}$ on $\mathcal{D}(\boldsymbol{A})$ (see Lemma 2.2.6 and the discussion below it). We will characterize which of the poset relations R define a nucleus, i.e., a closure operator satisfying $\gamma_R(X) \circ \gamma_R(Y) \subseteq \gamma_R(X \circ Y)$ for all $X, Y \in \mathcal{D}(\boldsymbol{A})$.

First, note that $\{{\downarrow}x \mid x \in A\}$ is a join-dense set in $\mathcal{D}(\boldsymbol{A})$. Indeed, we have $X = \bigcup_{x\in X} {\downarrow}x$ for any downset $X \in \mathcal{D}(\boldsymbol{A})$. Thus Lemma 3.7.2 can be reformulated as follows.

LEMMA 3.8.4. *Let $\boldsymbol{A} = \langle A, \cdot, 0, 1, \leq \rangle$ be a pointed pomonoid and γ a closure operator on $\mathcal{D}(\boldsymbol{A})$ with a basis $\mathcal{B}$. Then the following are equivalent:*

1. *γ is a nucleus on $\mathcal{D}(\boldsymbol{A})$.*
2. *$C/{\downarrow}x$ and ${\downarrow}x\backslash C$ are γ-closed for all $x \in A$ and $C \in \mathcal{B}$.*

Let $\boldsymbol{A} = \langle A, \cdot, 0, 1, \leq \rangle$ be a pointed pomonoid and $\boldsymbol{B} = \langle B, \leq \rangle$ a poset. A poset relation $R \subseteq A \times B$ is said to be *nuclear* if for all $x, y \in A$ and $z \in B$, there exist subsets $x\mathbin{\backslash\!\!\backslash} z, z /\!\!/ y \subseteq B$ such that

$$x \cdot y \mathrel{R} z \quad \text{iff} \quad y \mathrel{R} x\mathbin{\backslash\!\!\backslash} z \quad \text{iff} \quad x \mathrel{R} z /\!\!/ y\,,$$

where $y \mathrel{R} x\mathbin{\backslash\!\!\backslash} z$ means that y is in the relation with all the elements from $x\mathbin{\backslash\!\!\backslash} z$. The meaning of $x \mathrel{R} z /\!\!/ y$ is defined analogously.

The poset relations which define a nucleus on $\mathcal{D}(\boldsymbol{A})$ are exactly those which are nuclear. Before we prove it, observe that the following hold for a poset relation $R \subseteq A \times B$, $Z \in \mathcal{D}(\boldsymbol{A})$, $x \in A$ and $z \in B$:

$${\downarrow}x\backslash Z = \{y \in A \mid (\exists z \in Z)(xy \leq z)\} = \{y \in A \mid xy \in Z\}\,.$$

$$({\uparrow}z)^{\lhd} = \{a \in A \mid (\forall u \in {\uparrow}z)(a \mathrel{R} u)\} = \{a \in A \mid a \mathrel{R} z\}\,.$$

$${\downarrow}x\backslash({\uparrow}z)^{\lhd} = \{y \in A \mid xy \mathrel{R} z\}\,.$$

LEMMA 3.8.5. *Let $\boldsymbol{A} = \langle A, \cdot, 0, 1, \leq \rangle$ be a pointed pomonoid, $\boldsymbol{B} = \langle B, \leq \rangle$ a poset and $R \subseteq A \times B$ a poset relation. The closure operator γ_R on $\mathcal{D}(\boldsymbol{A})$ is a nucleus iff R is nuclear.*

Proof. By Lemma 2.2.7, the collection $\mathcal{B} = \{({\uparrow}z)^{\lhd} \mid z \in B\}$ forms a basis of γ_R. So by Lemma 3.8.4 γ_R is a nucleus iff ${\downarrow}x\backslash({\uparrow}z)^{\lhd}$ and $({\uparrow}z)^{\lhd}/{\downarrow}x$ are γ_R-closed for all $x \in A$ and $z \in B$. A downset ${\downarrow}x\backslash({\uparrow}z)^{\lhd}$ is γ_R-closed iff there is a subset $\mathcal{X} \subseteq \mathcal{B}$ such that ${\downarrow}x\backslash({\uparrow}z)^{\lhd} = \bigcap \mathcal{X}$ where $\mathcal{X} = \{({\uparrow}c)^{\lhd} \mid c \in x\mathbin{\backslash\!\!\backslash} z\}$ for some subset $x\mathbin{\backslash\!\!\backslash} z \subseteq B$. Thus ${\downarrow}x\backslash({\uparrow}z)^{\lhd}$ is γ_R-closed iff the following equivalence holds:

$$y \in {\downarrow}x\backslash({\uparrow}z)^{\lhd} \text{ iff } y \in ({\uparrow}c)^{\lhd} \text{ for all } c \in x\mathbin{\backslash\!\!\backslash} z.$$

This can be equivalently transformed to

$$xy \mathrel{R} z \text{ iff } y \mathrel{R} c \text{ for all } c \in x\mathbin{\backslash\!\!\backslash} z.$$

The last equivalence is nothing but the part of the definition of a nuclear poset relation. One can prove analogously that $({\uparrow}z)^{\lhd}/{\downarrow}x$ is γ_R-closed iff R satisfies

$$xy \mathrel{R} z \text{ iff } x \mathrel{R} z /\!\!/ y.$$

□

Let $\boldsymbol{A}$ be an FL-algebra. Now we can employ Lemma 3.8.5 in order to define a suitable nucleus on $\mathcal{D}(\boldsymbol{A})$ for the construction of the Dedekind-MacNeille completion of $\boldsymbol{A}$. Consider the lattice order $\leq\ \subseteq A \times A$ on $\boldsymbol{A}$. This is clearly a poset relation which is nuclear since there are singletons $x \mathbin{\backslash\!\!\backslash} z = \{x\backslash z\}$ and $z \mathbin{/\!\!/} y = \{z/y\}$ for all $x, y, z \in A$ such that

$$xy \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y\,.$$

Thus $\gamma_{\leq}$ is a nucleus on $\mathcal{D}(\boldsymbol{A})$. The resulting FL-algebra $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is called the *Dedekind-MacNeille completion* of $\boldsymbol{A}$. It is clear that the lattice reduct of $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is complete because the lattice reduct of $\mathcal{D}(\boldsymbol{A})$ is complete and the image of any closure operator on a complete lattice is complete as well. Note also that the lattice reduct of $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is the well-known Dedekind-MacNeille completion of the lattice reduct of $\boldsymbol{A}$ because it consists of all downsets which are stable under taking all upper bounds and then all lower bounds. It only remains to show that there is an embedding $f\colon \boldsymbol{A} \to \gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$.

THEOREM 3.8.6. *Every* FL*-algebra* $\boldsymbol{A}$ *can be embedded into its Dedekind-MacNeille completion* $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$.

Proof. Consider the map $f\colon A \to \mathcal{D}(\boldsymbol{A})$ defined by $f(x) = {\downarrow}x$. By Lemma 3.8.3 f is an embedding of the $\vee$-free reduct of $\boldsymbol{A}$ into $\mathcal{D}(\boldsymbol{A})$. In order to invoke Lemma 3.7.6 we have to show that every ${\downarrow}x$ is $\gamma_{\leq}$-closed, i.e., $({\downarrow}x)^{\rhd\lhd} = {\downarrow}x$. Since the relation giving the Galois connection $^{\rhd}, {}^{\lhd}$ is the order $\leq$ on $\boldsymbol{A}$, we have

$$\begin{aligned} ({\downarrow}x)^{\rhd} &= \{a \in A \mid a \geq x\} = {\uparrow}x\,, \\ ({\uparrow}x)^{\lhd} &= \{a \in A \mid a \leq x\} = {\downarrow}x\,. \end{aligned}$$

Thus $f\colon A \to \gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is an embedding of the $\vee$-free reduct of $\boldsymbol{A}$ into the $\vee$-free reduct of $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ by Lemma 3.7.6. It remains to check that f preserves also $\vee$. We have to show that

$${\downarrow}(x \vee y) = f(x \vee y) = f(x) \vee_{\gamma_{\leq}} f(y) = \gamma_{\leq}({\downarrow}x \cup {\downarrow}y) = ({\downarrow}x \cup {\downarrow}y)^{\rhd\lhd}\,.$$

Note that

$$({\downarrow}x \cup {\downarrow}y)^{\rhd} = \{u \in A \mid u \geq x, u \geq y\} = {\uparrow}(x \vee y)\,.$$

Thus $({\downarrow}x \cup {\downarrow}y)^{\rhd\lhd} = {\downarrow}(x \vee y)$. $\square$

We have seen that every FL-algebra $\boldsymbol{A}$ can be embedded into its Dedekind-MacNeille completion, namely the FL-algebra $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$. However, the applicability of this statement depends also on whether the algebra $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ remains in the same variety from which $\boldsymbol{A}$ was taken. The next theorem shows that some properties are preserved by the construction of $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$.

THEOREM 3.8.7. *Let* $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. *Then the varieties* $\mathbb{FL}_{\mathrm{S}}$ *and* $\mathbb{RL}_{\mathrm{S}}$ *are both closed under the Dedekind-MacNeille completion. The same is true also for the classes of all* FL_{S}*-chains and* RL_{S}*-chains.*

Proof. The FL-algebra $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is an $\mathrm{FL_S}$-algebra (resp. $\mathrm{RL_S}$-algebra), if $\boldsymbol{A}$ is. Indeed, using Lemma 3.8.2, $\mathcal{D}(\boldsymbol{A})$ is an $\mathrm{FL_S}$-algebra (resp. $\mathrm{RL_S}$-algebra), if $\boldsymbol{A}$ is. Since any nucleus preserves properties from S and also the identity $1 = 0$ (see Lemma 3.7.4), $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is an $\mathrm{FL_S}$-algebra (resp. $\mathrm{RL_S}$-algebra) as well. The rest of the theorem follows from the fact that a nuclear retraction of a chain is again a chain. □

Although the above theorem holds also for $\mathrm{RL_S}$-algebras, note that if $\{\mathrm{o}\} \subseteq \mathrm{S}$ then $\mathbb{RL}_\mathrm{S}$ is the trivial variety containing only the trivial algebra. Thus the above theorem is trivially valid in this case. Similar observations can be made also in the following theorems within this section.

Unfortunately, the Dedekind-MacNeille completion does not preserve the identity axiomatizing the semilinear FL-algebras. Thus $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ need not be an FL^{ℓ}-algebra if $\boldsymbol{A}$ is. However, we can use the subdirect representation theorem in order to obtain at least some completion, although this need not be regular.

THEOREM 3.8.8. *Let* $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$. *Then every* $\mathrm{FL}^{\ell}_{\mathrm{S}}$*-algebra (resp.* $\mathrm{RL}^{\ell}_{\mathrm{S}}$*-algebra) can be embedded into a complete* $\mathrm{FL}^{\ell}_{\mathrm{S}}$*-algebra (resp.* $\mathrm{RL}^{\ell}_{\mathrm{S}}$*-algebra).*

Proof. Let $\boldsymbol{A}$ be an $\mathrm{FL}^{\ell}_{\mathrm{S}}$-algebra (resp. $\mathrm{RL}^{\ell}_{\mathrm{S}}$-algebra). By Theorem 2.1.5 $\boldsymbol{A}$ can be embedded into $\prod_{i\in I} \boldsymbol{A}_i$ where $\boldsymbol{A}_i$'s are subdirectly irreducible and homomorphic images of $\boldsymbol{A}$. Thus all $\boldsymbol{A}_i$'s belong to the semilinear variety generated by $\boldsymbol{A}$. Moreover, all $\boldsymbol{A}_i$'s have to be $\mathrm{FL_S}$-chains (resp. $\mathrm{RL_S}$-chains) by Proposition 3.6.2. Using Theorem 3.8.6, each $\boldsymbol{A}_i$ can be embedded into its Dedekind-MacNeille completion $\overline{\boldsymbol{A}}_i$. Thus $\boldsymbol{A}$ is embeddable into $\prod_{i\in I} \overline{\boldsymbol{A}}_i$ which clearly has a complete lattice reduct. □

Another question is whether the Dedekind-MacNeille completion of an involutive FL-algebra remains involutive. We have seen that properties like commutativity, integrality etc. were preserved stepwise in the construction of $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$, i.e., if $\boldsymbol{A}$ is an $\mathrm{FL_S}$-algebra for $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$ then also $\mathcal{D}(\boldsymbol{A})$ and $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ are $\mathrm{FL_S}$-algebras. On the other hand, if $\boldsymbol{A}$ is an involutive FL-algebra, $\mathcal{D}(\boldsymbol{A})$ need not be involutive. Nevertheless, after applying $\gamma_{\leq}$, the resulting completion becomes involutive. Let $X \in \mathcal{D}(\boldsymbol{A})$. We define two sets $X^{\sim} = \{\sim x \mid x \in X\}$ and $X^{-} = \{-x \mid x \in X\}$. The following lemma shows how the negations $\sim X$ and $-X$ in $\mathcal{D}(\boldsymbol{A})$ (thus also in $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$) are related to the sets $X^{\sim}$ and X^{-}.

LEMMA 3.8.9. *Let* $\boldsymbol{A}$ *be an involutive* FL*-algebra. Then we have for all* $X \in \mathcal{D}(\boldsymbol{A})$*:*

$$\sim X = X^{\sim\lhd} = X^{\rhd\sim} \quad \text{and} \quad -X = X^{-\lhd} = X^{\rhd -}\,.$$

Proof. We have

$$\begin{aligned} \sim X &= X\backslash{\downarrow}0 \\ &= \{y \in A \mid (\forall x \in X)(xy \leq 0)\} \\ &= \{y \in A \mid (\forall x \in X)(y \leq \sim x)\} \\ &= X^{\sim\lhd}\,. \end{aligned}$$

Also

$$\begin{aligned} {\sim}X &= \{y \in A \mid (\forall x \in X)(xy \leq 0)\} \\ &= \{y \in A \mid (\forall x \in X)(x \leq -y)\} \\ &= \{y \in A \mid -y \in X^{\rhd}\} \\ &= \{y \in A \mid {\sim}{-y} \in X^{\rhd\sim}\} = X^{\rhd\sim}. \end{aligned}$$

The second part of the lemma is proved analogously. □

Then it is easy to see that $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ is involutive if we start with an involutive FL-algebra $\boldsymbol{A}$ because for any $\gamma_{\leq}$-closed downset X we have $-{\sim}X = X^{\rhd\sim-\lhd} = X^{\rhd\lhd} = X$. Similarly, ${\sim}{-X} = X$.

THEOREM 3.8.10. *Let* $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{w}\}$ *and* $\boldsymbol{A}$ *an involutive* $\mathrm{FL_S}$*-algebra (resp.* $\mathrm{RL_S}$*-algebra). Then its Dedekind-MacNeille completion* $\gamma_{\leq}[\mathcal{D}(\boldsymbol{A})]$ *is an involutive* $\mathrm{FL_S}$*-algebra (resp.* $\mathrm{RL_S}$*-algebra) as well.*

Again note that for $\mathrm{RL_w}$-algebras the above theorem holds trivially because there is only one $\mathrm{RL_w}$-algebra, namely the trivial one.

Theorem 3.8.6 can be also used in order to derive a structural characterization for FL-algebras. Namely, each FL-algebra $\boldsymbol{A}$ is embeddable into a nuclear retraction of a downset monoid $\mathcal{D}(\boldsymbol{M})$ for a pointed pomonoid $\boldsymbol{M}$ (namely $\boldsymbol{M}$ is the pointed-pomonoid reduct of $\boldsymbol{A}$). Thus if $\boldsymbol{A}$ is an $\mathrm{FL_S}$-algebra, then $\boldsymbol{M}$ satisfies the properties from S (i.e., commutativity, contractivity, integrality and zero-boundedness). Moreover, $1 = 0$ holds in $\boldsymbol{M}$ if $\boldsymbol{A}$ is an $\mathrm{RL_S}$-algebra. Thus we obtain the following theorem.

THEOREM 3.8.11. *Let* $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$*. Then the variety* $\mathbb{FL}_\mathrm{S}$ *(resp.* $\mathbb{RL}_\mathrm{S}$*) equals the class of subalgebras of nuclear retractions of downset monoids* $\mathcal{D}(\boldsymbol{M})$*, where* $\boldsymbol{M}$ *is a pointed pomonoid (resp. pomonoid) satisfying the properties from* S.

In addition, the class of $\mathrm{FL_S}$*-chains (resp.* $\mathrm{RL_S}$*-chains) equals the class of subalgebras of nuclear retractions of downset monoids* $\mathcal{D}(\boldsymbol{M})$*, where* $\boldsymbol{M}$ *is a linearly ordered pointed pomonoid (resp. linearly ordered pomonoid) satisfying the properties from* S.

4 Completeness with respect to distinguished semantics

Each semilinear variety $\mathbb{L}$ of FL-algebras defines its corresponding consequence relation $\models_{\mathbb{L}}$. In this section we are going to discuss whether it is possible in some cases to replace the class of algebras $\mathbb{L}$ by a smaller class $\mathbb{K} \subseteq \mathbb{L}$ in such a way that $\models_{\mathbb{K}} = \models_{\mathbb{L}}$. It may happen that it is not possible to have this equality for a given $\mathbb{K}$. In that case we can consider whether this equality holds when the relations $\models_{\mathbb{K}}$, $\models_{\mathbb{L}}$ are restricted to certain subsets.

Now we define more precisely what we have meant above by the restriction of $\models_{\mathbb{K}}$ and $\models_{\mathbb{L}}$. We distinguish the following kinds of completeness properties of $\mathbb{L}$ with respect to $\mathbb{K}$:

1. *Strong completeness* (S$\mathbb{K}$C): $\models_{\mathbb{L}} = \models_{\mathbb{K}}$.

2. *Finite strong completeness* (FS$\mathbb{K}$C): for every *finite* set of identities $E \cup \{s = t\}$ we have $E \models_{\mathbb{L}} s = t$ iff $E \models_{\mathbb{K}} s = t$. In other words, $\mathbb{L}$ is generated as a quasivariety by $\mathbb{K}$.

3. *Completeness* ($\mathbb{K}$C): for every identity $s = t$ we have $\models_{\mathbb{L}} s = t$ iff $\models_{\mathbb{K}} s = t$. In other words, $\mathbb{L}$ is generated as a variety by $\mathbb{K}$.

Clearly S$\mathbb{K}$C implies FS$\mathbb{K}$C which implies $\mathbb{K}$C. However, none of the reverse implications holds generally.

We will be interested mainly in the first two completeness properties. Let us recall how to prove these properties from Chapter II. We always have $\models_{\mathbb{L}} \subseteq \models_{\mathbb{K}}$ because $\mathbb{K} \subseteq \mathbb{L}$. Moreover, $\models_{\mathbb{L}} = \models_{\mathbb{L}_{\mathrm{C}}}$ since $\mathbb{L}$ is semilinear, where $\mathbb{L}_{\mathrm{C}}$ denotes the class of all chains from $\mathbb{L}$. Thus in order to prove S$\mathbb{K}$C for $\mathbb{L}$, it suffices to show $\models_{\mathbb{K}} \subseteq \models_{\mathbb{L}_{\mathrm{C}}}$. Assume that $E \not\models_{\mathbb{L}_{\mathrm{C}}} s = t$. Then there is a nontrivial FL-chain $\boldsymbol{A}$ from $\mathbb{L}_{\mathrm{C}}$ and an evaluation $e\colon Fm \to A$ satisfying all identities from E but not $s = t$. We may assume that $\boldsymbol{A}$ is countable since our language is countable. More precisely, we can take just the countable subalgebra of $\boldsymbol{A}$ generated by the countable set

$$G = \{e(\tau) \in A \mid \tau \text{ is a subterm occurring in } E \cup \{s = t\}\}\,.$$

Consequently, if we prove that each countable nontrivial member of $\mathbb{L}_{\mathrm{C}}$ is embeddable into a member $\boldsymbol{B}$ of $\mathbb{K}$ via an embedding $f\colon \boldsymbol{A} \to \boldsymbol{B}$, we are done since the composition $f \circ e$ gives an evaluation into $\boldsymbol{B}$ which satisfies all the premises from E but not the conclusion $s = t$. Thus $E \not\models_{\mathbb{K}} s = t$.

THEOREM 4.0.12. *Let $\mathbb{L}$ be a semilinear variety of FL-algebras and $\mathbb{K}$ a subclass of $\mathbb{L}$. If every countable nontrivial chain from $\mathbb{L}$ is embeddable into a member of $\mathbb{K}$ then $\mathbb{L}$ has the strong completeness property with respect to $\mathbb{K}$.*

In order to prove that $\mathbb{L}$ has FS$\mathbb{K}$C, we have to modify the above method a little bit. Again, we assume that $E \not\models_{\mathbb{L}_{\mathrm{C}}} s = t$ but now for a finite set of identities $E \cup \{s = t\}$. Thus there is a nontrivial FL-chain $\boldsymbol{A}$ from $\mathbb{L}_{\mathrm{C}}$ and an evaluation $e\colon Fm \to A$ satisfying all identities from E but not $s = t$. Again we suppose that $\boldsymbol{A}$ is generated by the set G which is now finite. In order to obtain a counterexample in $\mathbb{K}$, it suffices to find a partial embedding $f\colon G \to B$ from the partial subalgebra $\boldsymbol{G}$ of $\boldsymbol{A}$ into a member $\boldsymbol{B}$ of $\mathbb{K}$. Then one can define an evaluation $e'\colon Fm \to B$ by setting $e'(p) = f(e(p))$, if p is a variable occurring in $E \cup \{s = t\}$, and $e'(p)$ is arbitrary, if p is a variable not occurring in $E \cup \{s = t\}$. It is easy to see that the resulting evaluation e' satisfies all the identities from E but not $s = t$. Thus $E \not\models_{\mathbb{K}} s = t$. Summing up, if we want to prove that $\mathbb{L}$ has FS$\mathbb{K}$C, it is sufficient to show that every countable nontrivial chain $\boldsymbol{A} \in \mathbb{L}_{\mathrm{C}}$ is partially embeddable into $\mathbb{K}$ (see Definition 2.1.11).

THEOREM 4.0.13. *Let $\mathbb{L}$ be a semilinear variety of FL-algebras and $\mathbb{K}$ a subclass of $\mathbb{L}$. If every nontrivial chain from $\mathbb{L}$ is partially embeddable into $\mathbb{K}$ then $\mathbb{L}$ has the finite strong completeness property with respect to $\mathbb{K}$.*

When proving completeness properties for a semilinear variety $\mathbb{L}$, we will focus on the following subclasses of $\mathbb{L}$:

1. $\mathbb{Q}$ consists of all countably infinite dense chains from $\mathbb{L}$,

2. $\mathbb{R}$ is the class of all complete chains $\boldsymbol{C}$ from $\mathbb{L}$ containing a countably infinite subset S dense in $\boldsymbol{C}$ (i.e., if $x, y \in C$ such that $x < y$ then there is $s \in S$ such that $x < s < y$),

3. $\mathbb{F}$ is the class of all finite chains from $\mathbb{L}$.

The reason why we are interested in the above-mentioned types of classes comes from the following considerations. The motivation for the second type comes from fuzzy logic since fuzzy logics are usually understood as logics whose intended set of truth values is the real unit interval $[0,1]$. This corresponds exactly to the second type because each complete chain, containing a countably infinite subset dense in it, is order-isomorphic to $[0,1]$. Thus each algebra from $\mathbb{R}$ is isomorphic to an algebra whose universe is $[0,1]$. Note however that $[0,1]$ plays here only the role of an order type. Thus its bounds $0,1$ need not correspond with the interpretations of constants $0,1$ from $\mathcal{L}_{\mathrm{FL}}$, i.e., there are FL-chains $\boldsymbol{A} = \langle [0,1], \cdot, \backslash, /, \wedge, \vee, 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}} \rangle$ such that $0 < 0^{\boldsymbol{A}}$ and $1^{\boldsymbol{A}} < 1$. Nevertheless, if $\boldsymbol{A}$ is an $\mathrm{FL_i}$-chain, then $1^{\boldsymbol{A}} = 1$. Analogously, $0^{\boldsymbol{A}} = 0$ if $\boldsymbol{A}$ is an $\mathrm{FL_o}$-chain. Moreover, we have both if $\boldsymbol{A}$ is an $\mathrm{FL_w}$-chain.

On the other hand, the first type corresponds to the semantics whose set of truth values forms an interval in the set of rational numbers. The first type will serve as an intermediate step when proving a completeness property with respect to the second type. Finally, the last type is motivated mainly by an application point of view, namely if $\mathbb{L}$ has $\mathrm{FS}\mathbb{F}\mathrm{C}$ then the set of quasi-identities valid in $\mathbb{L}$ is decidable. Similarly, $\mathrm{S}\mathbb{F}\mathrm{C}$ implies decidability of the equational theory of $\mathbb{L}$. Note that $\mathrm{FS}\mathbb{F}\mathrm{C}$ is usually called the *finite embeddability property* (FEP) in the literature.

Concerning the structure of the following text on the completeness properties, we will split it into two parts. The first part contains positive results on completeness properties for integral semilinear varieties (i.e., semilinear varieties containing $\mathrm{FL_i}$-algebras). The second part deals with non-integral varieties where we have mainly negative results.

4.1 Integral semilinear varieties

4.1.1 Completeness w.r.t. countably infinite dense chains and $[0,1]$-valued semantics

Let $\langle A, \leq \rangle$ be a chain and $a, b \in A$. We denote the fact that a is a *subcover* of b as $a \prec b$, i.e., $a \prec b$ holds iff $a < b$ and there is no $c \in A$ such that $a < c < b$. A chain $\langle A, \leq \rangle$ is said to be *dense* if $a \prec b$ does not hold for any $a, b \in A$. Note that the trivial one element chain is dense by this definition.

Let $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$ and $\mathbb{FL}^{\ell}_{\mathrm{S}}$ the corresponding variety of semilinear $\mathrm{FL_S}$-algebras. First, we will focus on $\mathrm{S}\mathbb{Q}\mathrm{C}$ for $\mathbb{FL}^{\ell}_{\mathrm{S}}$. It is obvious that each chain from $\mathbb{Q}$ is isomorphic to an algebra whose universe is either the set $\mathsf{Q} \cap [0,1]$ or $\mathsf{Q} \cap (0,1]$ depending on the existence of a bottom element.

By Theorem 4.0.12 in order to show $\mathrm{S}\mathbb{Q}\mathrm{C}$ for $\mathbb{FL}^{\ell}_{\mathrm{S}}$, it is sufficient to prove that each countable nontrivial $\mathrm{FL_S}$-chain $\boldsymbol{A}$ can be embedded into a countably infinite dense $\mathrm{FL_S}$-chain $\boldsymbol{D}$. Suppose that we have an $\mathrm{FL_S}$-chain $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}} \rangle$ which is countable and nontrivial. If $\boldsymbol{A}$ is dense then A has to be infinite because $\boldsymbol{A}$ is nontrivial.

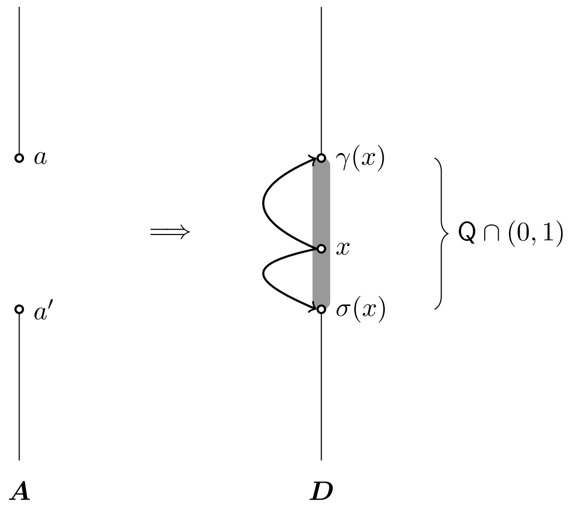

Figure 5. Filling gaps.

Thus we can set $\boldsymbol{D} = \boldsymbol{A}$. If $\boldsymbol{A}$ is not dense then there is at least one element a which has a subcover a'. As we want to extend $\boldsymbol{A}$ so that it becomes dense, we have to fill for each such element a the gap between a and a' by a countable dense chain. This can be done by pasting a copy of rational numbers (namely $\mathsf{Q} \cap (0, 1)$) into the gap between a and a' (see Figure 5). Formally we can define the set D as the following subset of $A \times (\mathsf{Q} \cap (0, 1])$:

$$D = \{\langle a, 1\rangle \mid a \in A\} \cup \{\langle a, q\rangle \mid q \in \mathsf{Q} \cap (0, 1) \text{ and } (\exists a' \in A) \text{ such that } a' \prec a\}\,. \quad (14)$$

Then the lexicographic order $\leq_{\text{lex}}$ on D is a dense linear order and $\langle 1^{\boldsymbol{A}}, 1\rangle$ is a top element. If $0^{\boldsymbol{A}}$ is a bottom element of $\boldsymbol{A}$ then $\langle 0^{\boldsymbol{A}}, 1\rangle$ is a bottom element of D because $0^{\boldsymbol{A}}$ has no subcover in this case. Moreover, the subset $A \times \{1\} \subseteq D$ is order-isomorphic to A.

Observe that we can define two operators on the chain $\boldsymbol{D} = \langle D, \leq_{\text{lex}}\rangle$ whose image is $A \times \{1\}$, namely a closure operator γ and an interior operator σ defined as follows:

$$\gamma(a, q) \quad = \quad \langle a, 1\rangle\,, \quad (15)$$

$$\sigma(a, q) \quad = \quad \begin{cases} \langle a, 1\rangle & \text{if } q = 1, \\ \langle a', 1\rangle & \text{if } q < 1 \text{ and } a' \prec a. \end{cases} \quad (16)$$

Note that both operators behave like the identity map on $A \times \{1\}$. Their behavior on the remaining elements is depicted in Figure 5.

It is easy to see that γ is a closure operator on $\boldsymbol{D} = \langle D, \leq_{\text{lex}}\rangle$. To see that σ is an interior operator on $\boldsymbol{D}$, note that σ is idempotent and contracting. Thus it suffices to show that σ is monotone. Assume that $\langle a, q\rangle \leq_{\text{lex}} \langle b, p\rangle$. If $p = 1$ then $\sigma(a, q) \leq_{\text{lex}} \langle a, q\rangle \leq_{\text{lex}} \langle b, 1\rangle = \sigma(b, 1)$. If $p < 1$ then there is $b' \in A$ such that $b' \prec b$. Now there are two cases. First, if $a < b$ then $a \leq b'$. Consequently, $\sigma(a,q) \leq_{\mathrm{lex}} \langle a,q\rangle \leq_{\mathrm{lex}} \langle b',1\rangle = \sigma(b,p)$. Second, if $a = b$ then $q \leq p$ and so $\sigma(a,q) = \langle b',1\rangle = \sigma(b,p)$. Thus σ is really an interior operator. Summing up, if we identify A with $A \times \{1\}$, we obtain following general lemma.

LEMMA 4.1.1. *Let $\boldsymbol{A} = \langle A, \wedge, \vee, 0, 1\rangle$ be a countable nontrivial chain with a top element* 1. *Then $\boldsymbol{A}$ can be extended to a countably infinite dense chain $\boldsymbol{D} = \langle D, \wedge, \vee, 0, 1\rangle$ with a top element* 1. *Moreover,* 0 *is a bottom element of $\boldsymbol{D}$ if* 0 *is a bottom element of $\boldsymbol{A}$. There are also a closure operator γ and an interior operator σ on $\boldsymbol{D}$ such that $A = \gamma[A] = \sigma[A]$.*

We have the dense chain $\boldsymbol{D}$. The next step is to extend the multiplication on the $\mathrm{FL_S}$-chain $\boldsymbol{A}$ to $\boldsymbol{D}$. In the light of Lemma 4.1.1 we will do this more generally, namely we do not restrict ourselves to the particular set D defined by (14) but we will consider a more abstract setting where we have a chain $\boldsymbol{D} = \langle D, \wedge, \vee, 0, 1\rangle$ together with constants $0, 1$ endowed with a closure operator γ and an interior operator σ such that $\gamma[D] = \sigma[D]$ and $\gamma[D]$ forms an $\mathrm{FL_S}$-chain $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$, i.e., $A = \gamma[D]$. Note that in this setting 1 has to be a top element in $\boldsymbol{D}$. Indeed, if there would be $x \in D$ such that $1 < x$ then $1 < \gamma(x) \in A$ which would mean that $\boldsymbol{A}$ is not integral. Also, if 0 is a bottom element of $\boldsymbol{A}$ then 0 has to be a bottom element in $\boldsymbol{D}$ because $x < 0$ implies $0 > \sigma(x) \in A$.

The first natural idea to extend the multiplication from $\boldsymbol{A}$ to the whole of $\boldsymbol{D}$, is to define the multiplication on $\boldsymbol{D}$ as $x \circ y = \gamma(x) \cdot \gamma(y)$, i.e., an element x which does not belong to A behaves like $\gamma(x)$. This works only partially since $1 \circ x = \gamma(1) \cdot \gamma(x) = 1 \cdot \gamma(x) = \gamma(x)$, i.e., 1 is not a neutral element of $\circ$. However, we still have the following result.

LEMMA 4.1.2. *Let $\boldsymbol{C} = \langle C, \wedge, \vee\rangle$ be a chain, γ a closure operator on $\boldsymbol{C}$ and σ an interior operator on $\boldsymbol{C}$ such that $\gamma[C] = \sigma[C]$. Further assume that the image $\gamma[C]$ forms a residuated lattice-ordered semigroup $\gamma[\boldsymbol{C}] = \langle \gamma[C], \cdot, \backslash, /, \wedge, \vee\rangle$. Then the algebra $\boldsymbol{C} = \langle C, \circ, \backslash^{\boldsymbol{C}}, /^{\boldsymbol{C}}, \wedge, \vee\rangle$ is a residuated lattice-ordered semigroup, where*

$$x \circ y = \gamma(x) \cdot \gamma(y)\,, \quad x /^{\boldsymbol{C}} y = \sigma(x)/\gamma(y)\,, \quad x \backslash^{\boldsymbol{C}} y = \gamma(x)\backslash\sigma(y)\,.$$

Moreover, $\boldsymbol{C}$ is commutative if $\gamma[\boldsymbol{C}]$ is.

Proof. The operation $\circ$ is clearly associative since $\cdot$ has this property. Thus $\boldsymbol{C}$ is a semigroup. Moreover, $\boldsymbol{C}$ is clearly commutative if $\gamma[\boldsymbol{C}]$ is. Thus it suffices to prove that $\boldsymbol{C}$ is residuated. Suppose that $x \circ y = \gamma(x) \cdot \gamma(y) \leq z$. Since $\gamma(x) \cdot \gamma(y)$ is σ-open, we have $\gamma(x) \cdot \gamma(y) = \sigma(\gamma(x) \cdot \gamma(y)) \leq \sigma(z)$. Consequently, $x \leq \gamma(x) \leq \sigma(z)/\gamma(y) = z/^{\boldsymbol{C}}y$. Conversely, suppose that $x \leq z/^{\boldsymbol{C}}y = \sigma(z)/\gamma(y)$. Since $\sigma(z)/\gamma(y)$ is γ-closed, we have $\gamma(x) \leq \gamma(\sigma(z)/\gamma(y)) = \sigma(z)/\gamma(y)$. Consequently, $x \circ y = \gamma(x) \cdot \gamma(y) \leq \sigma(z) \leq z$. Analogously for the left division. Thus $/^{\boldsymbol{C}}$ and $\backslash^{\boldsymbol{C}}$ are the residuals of $\circ$. □

Applying Lemma 4.1.2 to our dense chain $\boldsymbol{D}$, we obtain a residuated lattice-ordered semigroup $\boldsymbol{D} = \langle D, \circ, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}, \wedge, \vee\rangle$. However, as we mentioned above, $\circ$ is not a monoid operation because 1 is not a neutral element. In particular, $1 \circ x = \gamma(x) \geq x$, i.e., the result of $1 \circ x$ could be greater than we need. Thus we have to further modify $\circ$. Note that the chain $\boldsymbol{D}$ also forms an $\mathrm{FL_{ci}}$-chain $\boldsymbol{D}_\wedge = \langle D, \wedge, \rightarrow, \wedge, \vee, 0, 1\rangle$, where

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \tag{17}$$

In fact, the minimum operation $\wedge$ on $\boldsymbol{D}$ is the greatest among all integral ℓ-monoid operations on $\boldsymbol{D}$. Thus it seems to be natural to lessen the values of $\circ$ by a combination with $\wedge$. Precisely, we define a new operation on $\boldsymbol{D}$, which will have all the necessary properties, as follows:

$$x * y = (x \circ y) \wedge x \wedge y\,.$$

The next two lemmas show that $*$ has the desired properties. In the following lemma when we have a groupoid operation on a semilattice we adopt a convention that the groupoid operation binds stronger than the semilattice operation.

LEMMA 4.1.3. *Let $\langle M, \wedge, \circ, 1\rangle$ be an algebra such that $\langle M, \wedge, 1\rangle$ is a meet-semilattice with a top element 1 and $\langle M, \circ\rangle$ a semigroup satisfying:*

1. *the operation $\circ$ distributes over $\wedge$, i.e., $x \circ (y \wedge z) = x \circ y \wedge x \circ z$, $(y \wedge z) \circ x = y \circ x \wedge z \circ x$,*
2. *$1 \circ x \wedge x = x \circ 1 \wedge x = x$.*

Then $\langle M, *, 1\rangle$ is a monoid, where $x * y = x \circ y \wedge x \wedge y$. Moreover, $*$ is commutative if $\circ$ is.

Proof. First, we check that 1 is a neutral element. We have $1 * x = (1 \circ x) \wedge 1 \wedge x = x$. Analogously $x * 1 = x$. Second we prove that $*$ is associative. We have

$$\begin{aligned} (x * y) * z &= (x \circ y \wedge x \wedge y) * z \\ &= (x \circ y \wedge x \wedge y) \circ z \wedge x \circ y \wedge x \wedge y \wedge z \\ &= x \circ y \circ z \wedge x \circ z \wedge y \circ z \wedge x \circ y \wedge x \wedge y \wedge z\,. \end{aligned}$$

Similarly,

$$\begin{aligned} x * (y * z) &= x * (y \circ z \wedge y \wedge z) \\ &= x \circ (y \circ z \wedge y \wedge z) \wedge x \wedge y \circ z \wedge y \wedge z \\ &= x \circ y \circ z \wedge x \circ y \wedge x \circ z \wedge x \wedge y \circ z \wedge y \wedge z\,. \end{aligned}$$

Since $\wedge$ is commutative, we get $(x * y) * z = x * (y * z)$. Consequently $*$ is a monoid operation on M. The last claim about commutativity follows immediately from the definition of $*$. □

LEMMA 4.1.4. *Let $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, $\boldsymbol{D}_\wedge = \langle D, \wedge, \rightarrow, \wedge, \vee, 0, 1\rangle$ an $\mathrm{FL_{ci}}$-chain and γ a closure operator on $\boldsymbol{D}_\wedge$ such that $\gamma[\boldsymbol{D}] = \langle \gamma[D], \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ is an $\mathrm{FL_S}$-chain. Further, assume that σ is an interior operator on $\boldsymbol{D}_\wedge$ such that $\sigma[D] = \gamma[D]$.*

Then the algebra $\boldsymbol{D} = \langle D, *, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}, \wedge, \vee, 0, 1\rangle$ *is an* $\mathrm{FL_S}$*-chain where*

$$\begin{aligned} x * y &= \gamma(x) \cdot \gamma(y) \wedge x \wedge y\,, \\ x\backslash^{\boldsymbol{D}} y &= \gamma(x)\backslash\sigma(y) \vee x \to y\,, \\ x/^{\boldsymbol{D}} y &= \sigma(x)/\gamma(y) \vee y \to x\,. \end{aligned}$$

Moreover, the operations $*, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}$*, when restricted to* $\gamma[D]$*, coincide respectively with* $\cdot, \backslash, /$*, i.e.,* $\gamma[\boldsymbol{D}]$ *is a subalgebra of* $\boldsymbol{D}$*.*

Proof. Let $\circ$ denote the operation on D defined by $x \circ y = \gamma(x) \cdot \gamma(y)$. By Lemma 4.1.2 this operation is associative and monotone because it is residuated. Since D is a chain, the monotonicity of $\circ$ is equivalent to its distributivity over $\wedge$. Moreover, since $1 \in \gamma[D]$, we have $\gamma(1) = 1$. Thus $1 \circ x = \gamma(1) \cdot \gamma(x) = 1 \cdot \gamma(x) = \gamma(x) \geq x$. Similarly, $x \circ 1 \geq x$. Consequently, Lemma 4.1.3 applies to $*$ since $x * y = x \circ y \wedge x \wedge y$. Thus $*$ is a monoid operation on D.

Further, it follows from Lemma 4.1.2 that $\circ$ is residuated with the divisions given by $\gamma(x)\backslash\sigma(y)$ and $\sigma(x)/\gamma(y)$. We have to show that the join combination of these divisions with $\to$ gives the divisions for $*$. We have the following chain of equivalences:

$$\begin{aligned} x * y = x \circ y \wedge x \wedge y \leq z \quad &\text{iff} \quad x \circ y \leq z \text{ or } x \wedge y \leq z \\ &\text{iff} \quad x \leq \sigma(z)/\gamma(y) \text{ or } x \leq y \to z \\ &\text{iff} \quad x \leq \sigma(z)/\gamma(y) \vee y \to z = z/^{\boldsymbol{D}} y\,. \end{aligned}$$

Analogously we obtain $x * y \leq z$ iff $y \leq x\backslash^{\boldsymbol{D}} z$. Thus $\boldsymbol{D}$ forms an FL-chain. It is even an $\mathrm{FL_S}$-chain because 1 is the top element and 0 is the bottom element of $\boldsymbol{D}$ if it is a bottom element of $\gamma[\boldsymbol{D}]$ (see the discussion after Lemma 4.1.1). Further, $\boldsymbol{D}$ is clearly commutative if $\gamma[\boldsymbol{D}]$ is. Finally, if $\gamma[\boldsymbol{D}]$ is contractive (i.e., $x \circ y = \gamma(x) \wedge \gamma(y)$) then

$$x * y = \gamma(x) \wedge \gamma(y) \wedge x \wedge y = x \wedge y\,.$$

Thus $\boldsymbol{D}$ is contractive as well.

Finally, we have to show that the restrictions of operations $*, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}$ on $\gamma[D]$ behave like $\cdot, \backslash, /$. Since $\gamma[\boldsymbol{D}]$ forms an $\mathrm{FL_i}$-chain, we have $\gamma(x) \cdot \gamma(y) \leq \gamma(x) \wedge \gamma(y)$. Thus

$$\gamma(x) * \gamma(y) = \gamma(x) \cdot \gamma(y) \wedge \gamma(x) \wedge \gamma(y) = \gamma(x) \cdot \gamma(y)\,.$$

For the right division we have

$$\gamma(x)/^{\boldsymbol{D}}\gamma(y) = \sigma(\gamma(x))/\gamma(\gamma(y)) \vee \gamma(y) \to \gamma(x) = \gamma(x)/\gamma(y) \vee \gamma(y) \to \gamma(x)\,.$$

Using again that $\gamma[\boldsymbol{D}]$ forms an $\mathrm{FL_i}$-chain, we obtain

$$(\gamma(y) \to \gamma(x)) \cdot \gamma(y) \leq (\gamma(y) \to \gamma(x)) \wedge \gamma(y) = \gamma(y) \wedge (\gamma(y) \to \gamma(x)) \leq \gamma(x)\,.$$

Thus by residuation $\gamma(y) \to \gamma(x) \leq \gamma(x)/\gamma(y)$. Consequently, we have $\gamma(x)/^{\boldsymbol{D}}\gamma(y) = \gamma(x)/\gamma(y)$. Similarly, we can prove $\gamma(x)\backslash^{\boldsymbol{D}}\gamma(y) = \gamma(x)\backslash\gamma(y)$. $\square$

Now using Lemma 4.1.1 together with Lemma 4.1.4 we can prove that every countable nontrivial $\mathrm{FL_S}$-chain $\boldsymbol{A}$ (where $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$) is embeddable into a countably infinite dense $\mathrm{FL_S}$-chain $\boldsymbol{D}$. Moreover, it is clear that the construction of $\boldsymbol{D}$ preserves the identity $1 = 0$. Thus the next theorem follows by Theorem 4.0.12.

THEOREM 4.1.5. *Let* $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$. *Then the variety* $\mathbb{FL}_\mathrm{S}^\ell$ *has the strong completeness property with respect to the class of all countably infinite dense* $\mathrm{FL_S}$-*chains. The same is true also for the variety* $\mathbb{RL}_\mathrm{S}^\ell$ *if* $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i}\}$.

Thus we have proved $\mathrm{S\mathbb{Q}C}$ for $\mathbb{FL}_\mathrm{S}^\ell$. Now it is easy to extend this result to $\mathrm{S\mathbb{R}C}$ using the Dedekind-MacNeille completion. Let $\boldsymbol{A}$ be a countably infinite dense $\mathrm{FL_S}$-chain for $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$. Then $\boldsymbol{A}$ embeds into its Dedekind-MacNeille completion $\boldsymbol{A}'$ which is an $\mathrm{FL_S}$-chain as well (see Theorems 3.8.6 and 3.8.7). Then A' is order-isomorphic to $[0,1]$ because A' is a complete chain containing a countably infinite subset A dense in A' (the density follows from the well-known properties of the Dedekind-MacNeille completion). The same procedure can be done also with any countably infinite dense $\mathrm{RL_S}$-chain for $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i}\}$. Thus we get the following theorem.

THEOREM 4.1.6. *Let* $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$. *Then the variety* $\mathbb{FL}_\mathrm{S}^\ell$ *has the strong completeness property with respect to the class of all* $\mathrm{FL_S}$-*chains on* $[0,1]$. *The same is true also for the variety* $\mathbb{RL}_\mathrm{S}^\ell$ *if* $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i}\}$.

4.1.2 Involutive algebras

So far we have not discussed the completeness properties for involutive FL-algebras. We still assume that our involutive FL-algebras are integral. Recall that involutive $\mathrm{FL_i}$-algebras are in fact involutive $\mathrm{FL_w}$-algebras since integrality in this case implies that 0 is a bottom element. Hence, in the remaining part of this section we will work with involutive $\mathrm{FL_S}$-algebras where $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{w}\}$. The contractivity (c) is omitted since the only involutive $\mathrm{FL_{cw}}$-chain is the two element Boolean algebra. Thus the class of involutive $\mathrm{FL}_\mathrm{cw}^\ell$-algebras is the variety of Boolean algebras which has neither $\mathbb{Q}\mathrm{C}$ nor $\mathbb{R}\mathrm{C}$. It also makes no sense to deal with involutive $\mathrm{RL_w}$-algebras because there is only the trivial $\mathrm{RL_w}$-algebra.

Before we prove that the variety $\mathbb{InFL}_\mathrm{S}^\ell$ enjoys $\mathrm{S\mathbb{Q}C}$, we present a general lemma showing how to modify an $\mathrm{FL_w}$-chain so that it becomes involutive.

LEMMA 4.1.7. *Let* $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ *be an* $\mathrm{FL_w}$-*chain and* $\langle \sim, -\rangle$ *an involutive pair on* $\boldsymbol{A}$ *satisfying* $y \backslash\!\!\backslash {-x} = {\sim y} /\!\!/ x$ *for all* $x, y \in A$, *where*

$$x \backslash\!\!\backslash y = x\backslash y \vee {\sim x}\,, \qquad x /\!\!/ y = x/y \vee -y\,.$$

Then $\boldsymbol{A}_i = \langle A, \circ, \backslash\!\!\backslash, /\!\!/, \wedge, \vee, 0, 1\rangle$ *is an involutive* $\mathrm{FL_w}$-*chain, where* $x \circ y = -(y \backslash\!\!\backslash {\sim x})$. *Moreover, if* $\boldsymbol{A}$ *is commutative and* $\sim = -$ *then* $\boldsymbol{A}_i$ *is commutative as well.*

Proof. Note that $\sim$ and $-$ in this lemma do not denote the negations in $\boldsymbol{A}$. Further observe that ${\sim 1} = 0 = -1$ because $\sim$ and $-$ are order-reversing bijections, 1 is a top element and 0 a bottom element. Using Theorem 3.2.2, it suffices to show that $\boldsymbol{A}_i$ is an associative involutive division lattice whose unit is 1.

First, we show that (4) holds for $\mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}$. Since $\boldsymbol{A}$ is a chain, we have

$$\begin{aligned} y \leq x\mathbin{\backslash\!\!\backslash}z = x\backslash z \vee {\sim}x \quad &\text{iff} \quad y \leq x\backslash z \text{ or } y \leq {\sim}x \\ &\text{iff} \quad x \leq z/y \text{ or } x \leq -y \\ &\text{iff} \quad x \leq z/y \vee -y = z\mathbin{/\!\!/}y\,. \end{aligned}$$

Second, using the assumption that $\langle\sim, -\rangle$ satisfies the contraposition law $y\mathbin{\backslash\!\!\backslash}{-x} = {\sim}y\mathbin{/\!\!/}x$, we can prove the associativity for $\mathbin{\backslash\!\!\backslash}$ and $\mathbin{/\!\!/}$. Observe that in any FL-chain the following distributive laws holds:

$$a\backslash(b \vee c) = a\backslash b \vee a\backslash c\,, \qquad (b \vee c)/a = b/a \vee c/a\,.$$

Thus we have the following chain of equations:

$$\begin{aligned} &x\mathbin{\backslash\!\!\backslash}(z\mathbin{/\!\!/}y) = x\backslash(z/y \vee -y) \vee {\sim}x = x\backslash(z/y) \vee x\backslash{-y} \vee {\sim}x = x\backslash(z/y) \vee x\mathbin{\backslash\!\!\backslash}{-y} \\ &= (x\backslash z)/y \vee {\sim}x\mathbin{/\!\!/}y = (x\backslash z)/y \vee {\sim}x/y \vee -y = (x\backslash z \vee {\sim}x)/y \vee -y = (x\mathbin{\backslash\!\!\backslash}z)\mathbin{/\!\!/}y\,. \end{aligned}$$

Finally, we check that 1 is the unit for $\mathbin{\backslash\!\!\backslash}$ and $\mathbin{/\!\!/}$. We have

$$1\mathbin{\backslash\!\!\backslash}x = 1\backslash x \vee {\sim}1 = x \vee 0 = x\,.$$

Similarly $x\mathbin{/\!\!/}1 = x$.

To see the moreover part, observe that if $\boldsymbol{A}$ is commutative then $x\backslash y = y/x$ for all $x, y \in A$. Assuming $\sim\, = -$, we obtain $x\mathbin{\backslash\!\!\backslash}y = x\backslash y \vee {\sim}x = y/x \vee -x = y\mathbin{/\!\!/}x$. Thus $\boldsymbol{A}_i$ is commutative because by the contraposition law $y\mathbin{\backslash\!\!\backslash}{-x} = {\sim}y\mathbin{/\!\!/}x$ we have

$$x \circ y = -(y\mathbin{\backslash\!\!\backslash}{\sim}x) = -(y\mathbin{\backslash\!\!\backslash}{-x}) = -({\sim}y\mathbin{/\!\!/}x) = -(x\mathbin{\backslash\!\!\backslash}{\sim}y) = y \circ x\,.$$

□

Let $\boldsymbol{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ be a nontrivial countable involutive $\mathrm{FL_S}$-chain where $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{w}\}$ and $\langle\sim, -\rangle$ the involutive pair given by the negations on $\boldsymbol{A}$. To prove $\mathrm{S}\mathbb{Q}\mathrm{C}$ for $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$, we have to embed $\boldsymbol{A}$ into a countably infinite dense involutive $\mathrm{FL_S}$-chain (see Theorem 4.0.12). We know from the previous section (see Lemmas 4.1.1 and 4.1.4) that $\boldsymbol{A}$ can be embedded into a countably infinite dense $\mathrm{FL_S}$-chain $\boldsymbol{D} = \langle D, *, \backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}, \wedge, \vee, 0, 1\rangle$ endowed with a closure operator γ and an interior operator σ such that $A = \gamma[D] = \sigma[D]$. However, $\boldsymbol{D}$ need not be involutive. Namely, the negations in $\boldsymbol{D}$ are computed as follows (note that $\gamma(0) = 0 = \sigma(0)$ and $x \to 0 = 0$ for $x > 0$ and $0 \to 0 = 1$):

$$\begin{aligned} {\sim}^{\boldsymbol{D}}x &= x\backslash^{\boldsymbol{D}}0 = \gamma(x)\backslash\sigma(0) \vee x \to 0 = {\sim}\gamma(x) \vee x \to 0 = {\sim}\gamma(x)\,, \\ -^{\boldsymbol{D}}x &= 0/^{\boldsymbol{D}}x = \sigma(0)/\gamma(x) \vee x \to 0 = -\gamma(x) \vee x \to 0 = -\gamma(x)\,. \end{aligned}$$

Consequently, the negations do not form an involutive pair because

$$-^{\boldsymbol{D}}{\sim}^{\boldsymbol{D}}x = -\gamma({\sim}\gamma(x)) = -{\sim}\gamma(x) = \gamma(x) = {\sim}^{\boldsymbol{D}}-^{\boldsymbol{D}}x\,.$$

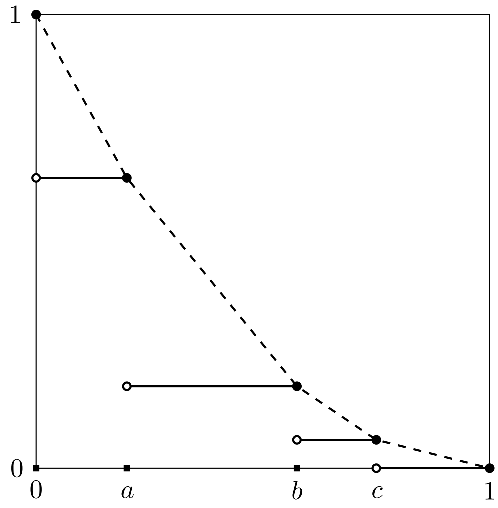

Figure 6. The involutive modification of $\sim$ in $\boldsymbol{D}$. The bold line denotes $\sim^{\boldsymbol{D}}$ and the dashed line its involutive modification $\sim^{\boldsymbol{D}_i}$.

The second equality follows from the fact that $\gamma[D]$ is closed under all operations from $\boldsymbol{A}$, in particular under $\sim$. An example of this situation is depicted in Figure 6, where $\{0, a, b, c, 1\}$ is the set of γ-closed elements.

In what follows we will modify both divisions $\backslash^{\boldsymbol{D}}, /^{\boldsymbol{D}}$ so that the corresponding negations become involutive and their restrictions on the γ-closed elements will behave like the original negations from $\boldsymbol{A}$ (see Figure 6). Recall that the dense chain $\boldsymbol{D}$ was defined in (14) as the set

$$D = \{\langle a, 1\rangle \mid a \in A\} \cup \{\langle a, q\rangle \mid q \in \mathsf{Q} \cap (0,1) \text{ and } (\exists a' \in A) \text{ such that } a' \prec a\}\,.$$

Inside D we identify A with $A \times \{1\}$ which is the image of the closure operator γ and the interior operator σ on $\boldsymbol{D}$ (see (15) and (16)). First, we show that it is possible to construct an involutive pair on $\boldsymbol{D}$ whose restrictions to the γ-closed and σ-open elements behave like the original negations $\sim, -$ from $\boldsymbol{A}$.

LEMMA 4.1.8. *Let $\sim^{\boldsymbol{D}_i}$ and $-^{\boldsymbol{D}_i}$ be unary operations on D defined as follows:*

$$\sim^{\boldsymbol{D}_i}\langle a, q\rangle = \begin{cases} \langle \sim a, 1\rangle & \text{if } q = 1,\\ \langle \sim a', 1-q\rangle & \text{if } q < 1,\end{cases}$$

$$-^{\boldsymbol{D}_i}\langle a, q\rangle = \begin{cases} \langle -a, 1\rangle & \text{if } q = 1,\\ \langle -a', 1-q\rangle & \text{if } q < 1,\end{cases}$$

where a' is the unique subcover of a. Then the pair $\langle \sim^{\boldsymbol{D}_i}, -^{\boldsymbol{D}_i}\rangle$ is an involutive pair such that $\sim^{\boldsymbol{D}_i}\langle a, 1\rangle = \langle \sim a, 1\rangle$ and $-^{\boldsymbol{D}_i}\langle a, 1\rangle = \langle -a, 1\rangle$. Moreover, $\sim^{\boldsymbol{D}_i} = -^{\boldsymbol{D}_i}$ if $\sim = -$.

Proof. Let $\langle a,q\rangle \le \langle b,r\rangle$. We show that ${\sim}^{\boldsymbol{D}_i}$ is order-reversing, i.e., ${\sim}^{\boldsymbol{D}_i}\langle b,r\rangle \le {\sim}^{\boldsymbol{D}_i}\langle a,q\rangle$. If $a = b$ then $q \le r$. Consequently, $1-r \le 1-q$. Thus ${\sim}^{\boldsymbol{D}_i}\langle b,r\rangle \le {\sim}^{\boldsymbol{D}_i}\langle a,q\rangle$ holds in this case. Now suppose that $a < b$. Depending on the values of q,r we have several cases. If $q < 1$ then $a' < b$ and $a' < b'$ if the subcover b' of b exists. Thus ${\sim}a' > {\sim}b$ and also ${\sim}a' > {\sim}b'$ if b' exists. Consequently, ${\sim}^{\boldsymbol{D}_i}\langle b,r\rangle \le {\sim}^{\boldsymbol{D}_i}\langle a,q\rangle$. If $q = r = 1$ then ${\sim}a > {\sim}b$ and the desired inequality again holds. The last possible case is $q = 1$ and $r < 1$. If $a < b'$ then the desired inequality follows as before. If $a = b'$ then ${\sim}^{\boldsymbol{D}_i}\langle b,r\rangle = \langle{\sim}a, 1-r\rangle < \langle{\sim}a,1\rangle = {\sim}^{\boldsymbol{D}_i}\langle a,q\rangle$. Similarly, one can prove that $-^{\boldsymbol{D}_i}$ is order-reversing.

Finally, we check that $-^{\boldsymbol{D}_i}{\sim}^{\boldsymbol{D}_i}\langle a,q\rangle = \langle a,q\rangle$. The proof of ${\sim}^{\boldsymbol{D}_i}-^{\boldsymbol{D}_i}\langle a,q\rangle = \langle a,q\rangle$ is analogous. If $q = 1$ then it clearly holds since $-^{\boldsymbol{D}_i}{\sim}^{\boldsymbol{D}_i}\langle a,1\rangle = \langle -{\sim}a,1\rangle = \langle a,q\rangle$. Suppose that $q < 1$. Then a has a unique subcover a', i.e., $a' \prec a$. Since $\sim$ is an order-reversing bijection, we have $x \prec y$ iff ${\sim}y \prec {\sim}x$. Thus ${\sim}a \prec {\sim}a'$, i.e., ${\sim}a$ is the unique subcover of ${\sim}a'$. Consequently, we obtain

$$-^{\boldsymbol{D}_i}{\sim}^{\boldsymbol{D}_i}\langle a,q\rangle = -\langle{\sim}a', 1-q\rangle = \langle -{\sim}a, 1-(1-q)\rangle = \langle a,q\rangle\,.$$

The moreover part of the lemma is obvious. □

Lemma 4.1.8 implies that there is an involutive pair $\langle{\sim}^{\boldsymbol{D}_i}, -^{\boldsymbol{D}_i}\rangle$ on the countably infinite dense $\mathrm{FL_S}$-chain $\boldsymbol{D}$ such that on γ-closed and σ-open elements coincides with the original negations from $\boldsymbol{A}$; recall the definitions of γ and σ from (15) and (16). Thus we have

$${\sim}^{\boldsymbol{D}_i}\gamma(x) = {\sim}\gamma(x)\,,\quad -^{\boldsymbol{D}_i}\gamma(x) = -\gamma(x)\,, \tag{18}$$

$${\sim}^{\boldsymbol{D}_i}\sigma(x) = {\sim}\sigma(x)\,,\quad -^{\boldsymbol{D}_i}\sigma(x) = -\sigma(x)\,. \tag{19}$$

Now we want to use the involutive pair $\langle{\sim}^{\boldsymbol{D}_i}, -^{\boldsymbol{D}_i}\rangle$ from the previous lemma in order to make $\boldsymbol{D}$ an involutive $\mathrm{FL_S}$-chain. In the light of Lemma 4.1.7 it is sufficient to show that the involutive pair $\langle{\sim}^{\boldsymbol{D}_i}, -^{\boldsymbol{D}_i}\rangle$ satisfies the contraposition law $y \mathbin{\backslash\!\!\backslash} -^{\boldsymbol{D}_i}x = {\sim}^{\boldsymbol{D}_i}y \mathbin{/\!\!/} x$, where

$$x \mathbin{\backslash\!\!\backslash} y = x\backslash^{\boldsymbol{D}}y \vee {\sim}^{\boldsymbol{D}_i}x\,,\qquad x \mathbin{/\!\!/} y = x/^{\boldsymbol{D}}y \vee -^{\boldsymbol{D}_i}y\,.$$

LEMMA 4.1.9. *We have* ${\sim}^{\boldsymbol{D}_i}\gamma(x) = \sigma({\sim}^{\boldsymbol{D}_i}x)$ *and* $-^{\boldsymbol{D}_i}\gamma(x) = \sigma(-^{\boldsymbol{D}_i}x)$*.*

Proof. Since $x \le \gamma(x)$, we have ${\sim}^{\boldsymbol{D}_i}\gamma(x) \le {\sim}^{\boldsymbol{D}_i}x$. Thus ${\sim}^{\boldsymbol{D}_i}\gamma(x) \le \sigma({\sim}^{\boldsymbol{D}_i}x)$ because ${\sim}^{\boldsymbol{D}_i}\gamma(x) = {\sim}\gamma(x)$ is σ-open. On the other hand, we have $\sigma({\sim}^{\boldsymbol{D}_i}x) \le {\sim}^{\boldsymbol{D}_i}x$. Hence $x \le -^{\boldsymbol{D}_i}\sigma({\sim}^{\boldsymbol{D}_i}x)$. Since $-^{\boldsymbol{D}_i}\sigma({\sim}^{\boldsymbol{D}_i}x) = -\sigma({\sim}^{\boldsymbol{D}_i}x)$ is γ-closed, we obtain $\gamma(x) \le -^{\boldsymbol{D}_i}\sigma({\sim}^{\boldsymbol{D}_i}x)$. Consequently, $\sigma({\sim}^{\boldsymbol{D}_i}x) \le {\sim}^{\boldsymbol{D}_i}\gamma(x)$. The other equality is proved analogously. □

LEMMA 4.1.10. *The involutive pair* $\langle{\sim}^{\boldsymbol{D}_i}, -^{\boldsymbol{D}_i}\rangle$ *satisfies the contraposition law*

$$y \mathbin{\backslash\!\!\backslash} -^{\boldsymbol{D}_i}x = {\sim}^{\boldsymbol{D}_i}y \mathbin{/\!\!/} x\,.$$

Proof. Recall from Lemma 4.1.4 that

$$y\backslash^{\boldsymbol{D}}x = \gamma(y)\backslash\sigma(x) \vee y \to x\,, \qquad y/^{\boldsymbol{D}}x = \sigma(y)/\gamma(x) \vee x \to y\,,$$

where $\to$ is the residual of $\wedge$ from $\boldsymbol{D}_\wedge$. Thus we have

$$\begin{aligned} y\mathbin{\backslash\!\!\backslash}-^{\boldsymbol{D}_i}x &= \gamma(y)\backslash\sigma(-^{\boldsymbol{D}_i}x) \vee (y \to -^{\boldsymbol{D}_i}x) \vee \sim^{\boldsymbol{D}_i}y\,,\\ \sim^{\boldsymbol{D}_i}y\mathbin{/\!\!/}x &= \sigma(\sim^{\boldsymbol{D}_i}y)/\gamma(x) \vee (x \to \sim^{\boldsymbol{D}_i}y) \vee -^{\boldsymbol{D}_i}x\,. \end{aligned}$$

Since $y \leq -^{\boldsymbol{D}_i}x$ iff $x \leq \sim^{\boldsymbol{D}_i}y$, we have $1 \leq y \to -^{\boldsymbol{D}_i}x$ iff $1 \leq x \to \sim^{\boldsymbol{D}_i}y$. Thus $y\mathbin{\backslash\!\!\backslash}-^{\boldsymbol{D}_i}x$ and $\sim^{\boldsymbol{D}_i}y\mathbin{/\!\!/}x$ both equal 1 if $y \leq -^{\boldsymbol{D}_i}x$ holds. Assume that $y > -^{\boldsymbol{D}_i}x$. Then also $x > \sim^{\boldsymbol{D}_i}y$. Consequently, $y \to -^{\boldsymbol{D}_i}x = -^{\boldsymbol{D}_i}x$ and $x \to \sim^{\boldsymbol{D}_i}y = \sim^{\boldsymbol{D}_i}y$. Thus it suffices to show that $\gamma(y)\backslash\sigma(-^{\boldsymbol{D}_i}x) = \sigma(\sim^{\boldsymbol{D}_i}y)/\gamma(x)$. By Lemma 4.1.9 and the equations (18), (19) we obtain

$$\gamma(y)\backslash\sigma(-^{\boldsymbol{D}_i}x) = \gamma(y)\backslash-^{\boldsymbol{D}_i}\gamma(x) = \gamma(y)\backslash-\gamma(x) = \sim\gamma(y)/\gamma(x) = \sigma(\sim^{\boldsymbol{D}_i}y)/\gamma(x)\,.$$

The last but one equality follows from the fact that $\langle\sim,-\rangle$ forms an involutive pair on $A = \gamma[D]$. Thus $\langle\sim,-\rangle$ has to satisfy the contraposition law $y\backslash-x = \sim y/x$. $\square$

Now using Lemma 4.1.7 together with Lemma 4.1.10, we can modify the countably infinite dense $\mathrm{FL_S}$-chain $\boldsymbol{D}$ so that $\boldsymbol{D}_i = \langle D, \circ, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}, \wedge, \vee, 0, 1\rangle$ becomes an involutive $\mathrm{FL_S}$-chain where

$$x\mathbin{\backslash\!\!\backslash}y = x\backslash^{\boldsymbol{D}}y \vee \sim^{\boldsymbol{D}_i}x\,, \qquad x\mathbin{/\!\!/}y = x/^{\boldsymbol{D}}y \vee -^{\boldsymbol{D}_i}y\,, \qquad x \circ y = -^{\boldsymbol{D}_i}(y\mathbin{\backslash\!\!\backslash}\sim^{\boldsymbol{D}_i}x)\,.$$

Now it remains to prove that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{D}_i$. Recall that $A = \gamma[D]$. We will show that the restrictions of $\mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}, \circ$ to $\gamma[D]$ coincide respectively with $\backslash, /, \cdot$. Let $\gamma(x), \gamma(y) \in \gamma[D]$. Since $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{D}$ (see Lemma 4.1.4), we obtain using also (18) the following:

$$\gamma(x)\mathbin{\backslash\!\!\backslash}\gamma(y) = \gamma(x)\backslash^{\boldsymbol{D}}\gamma(y) \vee \sim^{\boldsymbol{D}_i}\gamma(x) = \gamma(x)\backslash\gamma(y) \vee \sim\gamma(x) = \gamma(x)\backslash\gamma(y)\,.$$

The last equality holds because in any $\mathrm{FL_w}$-algebra we have $x\backslash y \geq x\backslash 0 = \sim x$. Similarly, we can prove $\gamma(x)\mathbin{/\!\!/}\gamma(y) = \gamma(x)/\gamma(y)$. Finally, since $\boldsymbol{A}$ is involutive, the multiplication in $\boldsymbol{A}$ is definable by $\backslash$, $\sim$ and $-$ as $u \cdot v = -(v\backslash\sim u)$. Thus using the fact that $\sim\gamma(x)$ and $\gamma(y)\backslash\sim\gamma(x)$ are γ-closed, we get using Lemma 3.2.1 and (18) the following chain of equations:

$$\gamma(x) \circ \gamma(y) = -^{\boldsymbol{D}_i}(\gamma(y)\mathbin{\backslash\!\!\backslash}\sim^{\boldsymbol{D}_i}\gamma(x)) = -(\gamma(y)\backslash\sim\gamma(x)) = \gamma(x)\cdot\gamma(y)\,.$$

Consequently, $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{D}_i$ and the next theorem follows.

THEOREM 4.1.11. *Let* $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{w}\}$. *Then the variety* $\mathbb{I}\mathrm{n}\mathbb{FL}_{\mathrm{S}}^{\ell}$ *has the strong completeness property with respect to the class of all countably infinite dense involutive* $\mathrm{FL_S}$-*chains.*

Using again the Dedekind-MacNeille completion (see Theorem 3.8.10), one can easily derive the following theorem.

THEOREM 4.1.12. *Let* $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e},\mathrm{w}\}$. *Then the variety* $\mathbb{I}\mathrm{n}\mathbb{FL}_{\mathrm{S}}^{\ell}$ *has the strong completeness property with respect to the class of all* $\mathrm{FL_S}$-*chains on* $[0,1]$.

4.1.3 Completeness w.r.t. finite algebras

Now we will turn our attention to the completeness properties with respect to the class of finite algebras. We will focus on the finite strong completeness property because one usually cannot hope for the strong completeness property in this case.

Before we start, we have to recall several results on dual well partial orders. Let $\boldsymbol{P}$ be a poset. We say that $\boldsymbol{P}$ is a *dual well partial order* (shortly *dwpo*) if $\boldsymbol{P}$ contains no infinite ascending chain and no infinite anti-chain. Dual well partial orders can be characterized by means of good sequences. A sequence $\vec{p} = \langle p_i \in P \mid i \in \mathsf{N}\rangle$ is said to be *good* if there are natural numbers $n < m$ such that $p_n \geq p_m$.

LEMMA 4.1.13 ([36]). *A poset $\boldsymbol{P}$ is a dwpo iff every infinite sequence of elements from P is good.*

The next two lemmas show how to construct new dwpos from given dwpos.

LEMMA 4.1.14. *The direct product of two dwpos is a dwpo.*

Let $\boldsymbol{P} = \langle P, \leq\rangle$ be a poset. By P^* we denote the set of all finite sequences of elements from P, i.e., P^* forms a free monoid generated by P. One can introduce a partial order $\sqsubseteq$ on P^* as follows:

$$\begin{aligned}&\langle a_0, \dots, a_m\rangle \sqsubseteq \langle b_0, \dots, b_n\rangle \text{ iff there exist } i_0, \dots, i_n \text{ such that}\\ &\ 0 \leq i_0 < i_1 < \cdots < i_n \leq m \text{ and } a_{i_k} \leq b_k \text{ for } k = 0, \dots, n.\end{aligned}$$

The following lemma is just a reformulation of the well-known result by Higman [24].

LEMMA 4.1.15 (Higman's lemma). *Let $\boldsymbol{P} = \langle P, \leq\rangle$ be a dwpo. Then $\boldsymbol{P}^* = \langle P^*, \sqsubseteq\rangle$ is a dwpo as well.*

For us it will be important to know that images of dwpos, which are linearly ordered, are dually well ordered as is proved in the following lemma.

LEMMA 4.1.16. *Let $\boldsymbol{P}$ be a dwpo, $\boldsymbol{Q}$ a chain and $f\colon P \to Q$ an order-preserving surjective map. Then $\boldsymbol{Q}$ is dually well ordered.*

Proof. Assume not. Then there is an ascending sequence $q_0 < q_1 < q_2 < \cdots$ of elements from Q. Take any sequence $\vec{p} = \langle p_i \in P \mid i \in \mathsf{N}\rangle$ such that $f(p_i) = q_i$. Since f is order-preserving, we have $p_n \not\geq p_m$ for any $n < m$. Thus $\vec{p}$ is not good which is not possible by Lemma 4.1.13. $\square$

We have presented all the necessary results on dwpos. Let $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. We are going to show that $\mathbb{FL}^\ell_{\mathrm{S}}$ enjoys FS$\mathbb{F}$C. In order to invoke Theorem 4.0.13 we have to show that every nontrivial FL_{S}-chain $\boldsymbol{A}$ is partially embeddable into the class of all finite FL_{S}-chains, i.e., for every finite partial subalgebra $\boldsymbol{G}$ of $\boldsymbol{A}$ we have to find a partial embedding from $\boldsymbol{G}$ into a finite FL_{S}-chain $\boldsymbol{B}$. We can assume without any loss of generality that $\boldsymbol{A}$ has a bottom element $\bot$. If not, then one can embed $\boldsymbol{A}$ into its Dedekind-MacNeille completion which has a bottom element because its lattice reduct is complete. We may also assume that G contains $\bot$, 0, and 1 because the restriction of a partial embedding $f\colon G \cup \{\bot, 0, 1\} \to B$ to G is a partial embedding as well.

Let $\boldsymbol{M}$ be the submonoid of $\boldsymbol{A}$ generated by G. Then $\boldsymbol{M}$ is clearly linearly ordered and contains $\bot, 0, 1$. Thus $\boldsymbol{M} = \langle M, \cdot, 0, 1, \leq \rangle$ is a pointed pomonoid with the bottom element $\bot$. Using Higman's lemma, we can prove the following lemma.

LEMMA 4.1.17. *The pointed pomonoid $\boldsymbol{M}$ is dually well ordered.*

Proof. The set G is finite and linearly ordered by the order $\leq$ on $\boldsymbol{A}$. Thus G forms a dwpo. By Higman's lemma $\langle G^*, \sqsubseteq \rangle$ is a dwpo as well. Moreover, G^* is a free monoid. Thus there is a surjective monoid homomorphism $h\colon G^* \to M$ such that $h(a_0, \dots, a_m) = a_0 \cdots a_m$. If we show that h is order-preserving, then the claim follows by Lemma 4.1.16. Suppose that $\langle a_0, \dots, a_m \rangle \sqsubseteq \langle b_0, \dots, b_n \rangle$, i.e., there are $i_0, \dots, i_n$ such that $0 \leq i_0 < i_1 < \cdots < i_n \leq m$ and $a_{i_k} \leq b_k$. Since h is a monoid homomorphism and $\boldsymbol{A}$ is integral, we have

$$h(a_0, \dots, a_m) = a_0 \cdots a_m \leq a_{i_0} \cdots a_{i_n} \leq b_0 \cdots b_n = h(b_0, \dots, b_n)\,.$$

□

It follows from Lemma 4.1.17 that M is the image of a conucleus σ on $\boldsymbol{A}$. Indeed, M induces an interior operator $\sigma(x) = \max\{a \in M \mid a \leq x\}$. The maximum always exists because M is dually well ordered and contains the bottom element $\bot$. Since M is a submonoid of $\boldsymbol{A}$, σ is a conucleus by Lemma 3.7.1. Consequently, $\boldsymbol{M}$ forms an FL-algebra, namely the σ-contraction $\sigma[\boldsymbol{A}] = \langle \sigma[A], \cdot, \backslash_\sigma, /_\sigma, \wedge, \vee, 0, 1 \rangle$, where $\sigma[A] = M$. Note that the constant 0 is not modified in $\sigma[\boldsymbol{A}]$ because $0 \in M$. The meet $\wedge$ need not be modified as well because $\boldsymbol{A}$ is linearly ordered. Indeed, let $x, y \in \sigma[A]$. Without any loss of generality we may assume that $x \leq y$. Thus $\sigma(x \wedge y) = \sigma(x) = x = x \wedge y$. Moreover, $\sigma[A]$ is an $\mathrm{FL_S}$-chain by Lemma 3.7.4.

Now using Lemma 3.7.5, there is a partial embedding $f\colon G \to \sigma[A]$ from $\boldsymbol{G}$ into $\sigma[\boldsymbol{A}]$ because $G \subseteq M = \sigma[A]$. The next step is to find a finite subset $B \subseteq \sigma[A]$ which forms an image of a nucleus γ on $\sigma[\boldsymbol{A}]$. Let B be the subset of $\sigma[A]$ defined as follows:

$$B = \{a \backslash_\sigma c /_\sigma b \in \sigma[A] \mid a, b \in \sigma[A],\ c \in G\} = \bigcup_{c \in G} \{a \backslash_\sigma c /_\sigma b \in \sigma[A] \mid a, b \in \sigma[A]\}\,.$$

In the above definition we are writing terms of the form $a \backslash_\sigma c /_\sigma b$ without parentheses because $(a \backslash c)/b = a \backslash (c/b)$ holds in any FL-algebra.

LEMMA 4.1.18. *The subset B is finite.*

Proof. Clearly, since G is finite, it is sufficient to prove that $B_c = \{a \backslash_\sigma c /_\sigma b \in \sigma[A] \mid a, b \in \sigma[A]\}$ is finite for a fixed $c \in G$. Assume that B_c is not finite. Since $\sigma[A] = M$ is dually well ordered, B_c has to contain an infinite descending chain $a_0 \backslash_\sigma c /_\sigma b_0 > a_1 \backslash_\sigma c /_\sigma b_1 > \cdots$. This chain defines a sequence $\vec{p} = \langle \langle a_i, b_i \rangle \in M^2 \mid i \in \mathsf{N} \rangle$. By Lemma 4.1.14 the direct product $\langle M^2, \leq \rangle$ forms a dwpo. Thus $\vec{p}$ has to be good, i.e., there are $n < m$ such that $\langle a_n, b_n \rangle \geq \langle a_m, b_m \rangle$. Consequently, we have

$$a_m (a_n \backslash_\sigma c /_\sigma b_n) b_m \leq a_n (a_n \backslash_\sigma c /_\sigma b_n) b_n \leq c\,.$$

Hence by residuation we obtain $a_n \backslash_\sigma c /_\sigma b_n \leq a_m \backslash_\sigma c /_\sigma b_m$; a contradiction with the fact that $a_0 \backslash_\sigma c /_\sigma b_0 > a_1 \backslash_\sigma c /_\sigma b_1 > \cdots$ is a descending chain. □

Now we have to show that the set B is the image of a nucleus γ. First, note that $G \subseteq B$ because $g = 1\backslash_\sigma g/_\sigma 1$ for every $g \in G$. In particular, $1 \in B$. Thus B induces a closure operator $\gamma(x) = \min\{a \in B \mid x \leq a\}$. The minimum always exists because B is finite and $1 \in B$ is the top element. In order to invoke Lemma 3.7.1 we have to show that $x/_\sigma y, y\backslash_\sigma x \in B$ for $x \in B$ and $y \in \sigma[A]$. Let $x \in B$ and $y \in \sigma[A]$. Then $x = a\backslash_\sigma c/_\sigma b$ for $c \in G$ and $a, b \in \sigma[A]$. Thus $x/_\sigma y = ((a\backslash_\sigma c)/_\sigma b)/_\sigma y = a\backslash_\sigma c/_\sigma yb$ by Lemma 3.0.12. Consequently, $x/_\sigma y \in B$. Analogously, $y\backslash_\sigma x \in B$. Thus B forms an FL-algebra, namely the γ-retraction $\gamma[\sigma[\boldsymbol{A}]]$. Moreover, $\gamma[\sigma[\boldsymbol{A}]]$ remains an $\mathrm{FL_S}$-chain by Lemma 3.7.4.

Since $G \subseteq B$, Lemma 3.7.6 implies that the partial embedding $f\colon G \to \sigma[A]$ is in fact a partial embedding of $\boldsymbol{G}$ into $\gamma[\sigma[\boldsymbol{A}]]$. Thus the next theorem follows. Note that the identity $1 = 0$ is preserved by σ and γ. Thus we get $\mathrm{FS\mathbb{F}C}$ also for $\mathbb{RL}^\ell_\mathrm{S}$.

THEOREM 4.1.19. *Let* $\{\mathrm{i}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$. *The variety* $\mathbb{FL}^\ell_\mathrm{S}$ *(resp.* $\mathbb{RL}^\ell_\mathrm{S}$*) has the finite strong completeness property with respect to the class of all finite* $\mathrm{FL_S}$*-chains (resp.* $\mathrm{RL_S}$*-chains).*

4.1.4 Cyclic involutive algebras

Now we are going to discuss whether the proof of Theorem 4.1.19 can be used also for involutive $\mathrm{FL}^\ell_\mathrm{i}$-algebras. Recall that involutive $\mathrm{FL_i}$-chains are in fact involutive $\mathrm{FL_w}$-chains and there is only one nontrivial involutive $\mathrm{FL_{cw}}$-chain, namely the two element Boolean algebra $\mathbf{2}$. Since the variety of Boolean algebras (i.e., the semilinear variety $\mathbb{I}\mathrm{n}\mathbb{FL}^\ell_\mathrm{cw}$) obviously enjoys the strong completeness property with respect to finite chains ($\mathrm{S\mathbb{F}C}$), we will focus only on involutive $\mathrm{FL_S}$-algebras for $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{w}\}$.

Let $\boldsymbol{A}$ be a nontrivial involutive $\mathrm{FL_S}$-chain where $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{w}\}$. The construction of the finite $\mathrm{FL_S}$-chain $\gamma[\sigma[\boldsymbol{A}]]$ from the proof of Theorem 4.1.19 need not preserve the double negation law $-{\sim}x = x = {\sim}{-}x$. Thus the above method cannot be used in order to prove that the variety of involutive $\mathrm{FL}^\ell_\mathrm{S}$-algebras has $\mathrm{FS\mathbb{F}C}$. Nevertheless the method works at least for cyclic involutive $\mathrm{FL}^\ell_\mathrm{S}$-algebras. Thus assume further that $\boldsymbol{A}$ is also cyclic, i.e., it satisfies $x\backslash 0 = {\sim}x = -x = 0/x$. Let $\boldsymbol{G}$ be a finite partial subalgebra of $\boldsymbol{A}$. Without any loss of generality we will assume that $\{0, 1\} \subseteq G$ (observe that 0 is a bottom element) and G is closed under the negations ${\sim}, -$. This does not affect finiteness of G because $\boldsymbol{A}$ is cyclic (i.e., ${\sim}x = -x$) and involutive (i.e., $-{\sim}x = x = {\sim}{-}x$).

We start with a general lemma on cyclic FL-algebras. Let $\boldsymbol{A}$ be an FL-algebra. We call an element $c \in A$ *involutive* if ${\sim}{-}c = c = -{\sim}c$. The next lemma shows that having an involutive element c in a cyclic FL-algebra $\boldsymbol{A}$, one can find other involutive elements in $\boldsymbol{A}$.

LEMMA 4.1.20. *Let* $\boldsymbol{A}$ *be a cyclic* FL*-algebra and* $c \in A$ *an involutive element. Then* $a\backslash c/b$ *is involutive as well for all* $a, b \in A$.

Proof. Since $\boldsymbol{A}$ is cyclic (i.e., ${\sim} = -$), it suffices to prove $-{\sim}(a\backslash c/b) = a\backslash c/b$. We first show that $a\backslash c/b = -(b(-c)a)$. Using Lemma 3.0.13, we obtain

$$-(b(-c)a) = -((-c)a)/b = {\sim}((-c)a)/b = a\backslash({\sim}{-}c)/b = a\backslash c/b\,.$$

Consequently, again using Lemma 3.0.13 and the latter equation, we have

$$-{\sim}(a\backslash c/b) = -{\sim}-(b(-c)a) = -(b(-c)a) = a\backslash c/b\,.$$

□

In the same way as in the proof of Theorem 4.1.22 we construct the finite algebra $\gamma[\sigma[\boldsymbol{A}]]$, where $\sigma[A]$ is the submonoid of $\boldsymbol{A}$ generated by G and

$$\gamma[\sigma[A]] = \{a\backslash_\sigma c/_\sigma b \in \sigma[A] \mid a, b \in \sigma[A],\ c \in G\}$$

Moreover, we know from the proof of Theorem 4.1.19 that there is a partial embedding of $\boldsymbol{G}$ into $\gamma[\sigma[\boldsymbol{A}]]$. Thus it suffices to prove that $\gamma[\sigma[\boldsymbol{A}]]$ is cyclic and involutive.

LEMMA 4.1.21. *The finite* FL_S*-chain* $\gamma[\sigma[\boldsymbol{A}]]$ *is cyclic and involutive.*

Proof. First, observe that $0 \in G \subseteq \gamma[\sigma[A]]$. Thus $\gamma[\sigma[\boldsymbol{A}]]$ is a subalgebra of $\sigma[\boldsymbol{A}]$ not only with respect to the language $\{\wedge, \backslash, /\}$ (see Lemma 3.7.4) but also $\{\wedge, \backslash, /, 0\}$. Thus the negations in $\gamma[\sigma[\boldsymbol{A}]]$ are just the restrictions of the negations from $\sigma[\boldsymbol{A}]$. Consequently, in order to check that $\gamma[\sigma[\boldsymbol{A}]]$ is cyclic and involutive, it suffices to show that $\sigma[\boldsymbol{A}]$ is cyclic and $\gamma[\sigma[A]]$ contains only involutive elements from $\sigma[\boldsymbol{A}]$. Let ${\sim}_\sigma$, $-_\sigma$ denote the negations in $\sigma[\boldsymbol{A}]$ (the symbols $\sim, -$ stand for the negations in $\boldsymbol{A}$). The cyclicity of $\sigma[\boldsymbol{A}]$ is easy because for $x \in \sigma[A]$ we have

$${\sim}_\sigma x = x\backslash_\sigma 0 = \sigma(x\backslash 0) = \sigma({\sim}x) = \sigma(-x) = \sigma(0/x) = 0/_\sigma x = -_\sigma x\,.$$

Thus $\sigma[\boldsymbol{A}]$ is cyclic. Consequently, $\gamma[\sigma[\boldsymbol{A}]]$ is cyclic.

To see that $\gamma[\sigma[\boldsymbol{A}]]$ is involutive, note that every $c \in G$ satisfies $-{\sim}c = c$ because $\boldsymbol{A}$ is involutive. Since G is closed under negations and all elements from G are σ-open, we have ${\sim}_\sigma c = \sigma({\sim}c) = {\sim}c$. Consequently, we obtain

$$-_\sigma{\sim}_\sigma c = \sigma(-{\sim}c) = \sigma(c) = c\,.$$

Thus every $c \in G$ is involutive in $\sigma[\boldsymbol{A}]$. Consequently, since every element from $\gamma[\sigma[A]]$ is of the form $a\backslash_\sigma c/_\sigma b$ for $c \in G$ and $a, b \in \sigma[A]$, all the elements from $\gamma[\sigma[A]]$ are involutive in $\sigma[\boldsymbol{A}]$ by Lemma 4.1.20. Thus $\gamma[\sigma[\boldsymbol{A}]]$ is involutive. □

THEOREM 4.1.22. *Let* $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{w}\}$. *The variety* $\mathbb{C}\mathrm{y}\mathbb{I}\mathrm{n}\mathbb{FL}^\ell_\mathrm{S}$ *of semilinear cyclic involutive* FL_S*-algebras has the finite strong completeness property with respect to the class of all finite cyclic involutive* FL_S*-chains.*

4.2 Non-integral semilinear varieties

Now we turn our attention to the semilinear varieties of FL-algebras which are not integral (i.e., they contain also non-integral FL-algebras). Unlike the previous part on integral varieties, we will present mainly negative results on completeness properties showing that most of the non-integral varieties we are discussing here do not enjoy the completeness properties with respect to the previously considered classes of algebras. As before we start with the completeness properties with respect to the class of all countably infinite dense chains and chains on $[0, 1]$.

4.2.1 Completeness w.r.t. countably infinite dense chains and $[0,1]$-valued semantics

It turns out that the density of an FL-chain $\boldsymbol{A}$ forces $\boldsymbol{A}$ to satisfy an extra identity which is not generally valid in FL-algebras. In particular, there are non-integral FL-chains where this identity does not hold.

LEMMA 4.2.1. *Every dense* FL*-chain* $\boldsymbol{A}$ *satisfies the following identity*

$$1 \leq (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x)))\,. \tag{20}$$

Proof. If $z \leq y$ then $1 \leq z\backslash y$ and the identity is clearly valid. Thus assume $y < z$ which is equivalent to $z\backslash y < 1$. Then we have to show that $1 \leq (x\backslash y)\backslash(1/(z\backslash x))$ which, by the residuation property, is equivalent to $(x\backslash y)\cdot(z\backslash x) \leq 1$. We will prove the latter inequality by reductio ad absurdum. Let $a = x\backslash y$ and $b = z\backslash x$. To get a contradiction, assume that $a \cdot b > 1$. Observe that by Lemma 3.0.12 we have $b \cdot a = (z\backslash x)\cdot(x\backslash y) \leq z\backslash y < 1$. This means that one of a, b is strictly greater than 1 and the other one strictly less. Without any loss of generality suppose that $a > 1$ and $b < 1$. Observe that b cannot be a bottom element $\bot$ because $\bot \cdot u = u \cdot \bot = \bot$ for all $u \in A$. Thus the set $M = \{u \in A \mid u < b\}$ is not empty. Moreover, since $\boldsymbol{A}$ is dense, we get $\bigvee M = b$. Consequently, $1 < a \cdot b = a \cdot \bigvee M = \bigvee_{u\in M}(a\cdot u)$. Thus there is $u' \in M$ such that $1 \leq a \cdot u'$. Now we have the following chain of inequalities:

$$b\cdot(a\cdot b) = (b\cdot a)\cdot b \leq b \leq b\cdot(a\cdot u') \leq b\cdot(a\cdot b)\,.$$

Thus $b\cdot(a\cdot u') = b$. On the other hand, $(b\cdot a)\cdot u' \leq u' < b$ which is a contradiction since $\cdot$ is associative. □

The above lemma shows that every dense chain has to satisfy the identity $1 \leq (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x)))$. In particular, it holds in every FL-chain whose universe is $[0,1]$. However, there is a four element FL-chain where this identity is not valid as is shown in the following lemma.

LEMMA 4.2.2. *There is an* $\mathrm{RL_c}$*-chain* $\boldsymbol{A}_4$ *and an* $\mathrm{FL_{co}}$*-chain* $\boldsymbol{B}_4$ *where*

$$1 \leq (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x)))$$

does not hold.

Proof. Let $A_4 = \{\bot, a, 1, \top\}$ ordered by $\bot < a < 1 < \top$ (see Figure 7). The multiplication $\cdot$ is defined as follows:

$\cdot$	$\bot$	a	1	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	$\top$
1	$\bot$	a	1	$\top$
$\top$	$\bot$	a	$\top$	$\top$

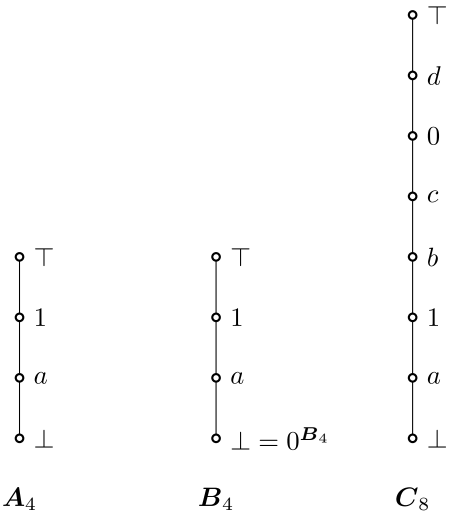

Figure 7. The algebras $\boldsymbol{A}_4$, $\boldsymbol{B}_4$ and $\boldsymbol{C}_8$.

It is straightforward to check that $\cdot$ is associative. Moreover, it is clearly residuated as A_4 is finite. Thus $\boldsymbol{A}_4 = \langle A_4, \cdot, \backslash, /, \wedge, \vee, 1\rangle$ is an RL-chain. Moreover, $\boldsymbol{A}_4$ is an $\mathrm{RL_c}$-chain because $\boldsymbol{A}_4$ is idempotent, i.e., $x = x^2$ holds. Let $x = \top$, $y = a$ and $z = 1$. Then $z\backslash y = a$, $x\backslash y = a$ and $z\backslash x = \top$. Consequently,

$$(z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x))) = a \vee (a\backslash(1/\top)) = a \vee (a\backslash\bot) = a \vee \bot = a < 1\,.$$

Thus $\not\models_{\boldsymbol{A}} 1 \leq (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x)))$. The $\mathrm{FL_{co}}$-chain $\boldsymbol{B}_4$ can be easily obtain from $\boldsymbol{A}_4$ by interpreting the constant 0 as $\bot$ (see Figure 7). $\square$

Let $\mathrm{S} \subseteq \{\mathrm{c}, \mathrm{o}\}$. Using Lemmas 4.2.1 and 4.2.2, it is clear that the variety $\mathbb{FL}^{\ell}_{\mathrm{S}}$ of semilinear $\mathrm{FL_S}$-algebras enjoy neither $\mathbb{Q}\mathrm{C}$ nor $\mathbb{R}\mathrm{C}$. The same holds for $\mathbb{RL}^{\ell}$ and $\mathbb{RL}^{\ell}_{\mathrm{c}}$. Thus we obtain the following theorem.

THEOREM 4.2.3. *Let* $\mathrm{S} \subseteq \{\mathrm{c}, \mathrm{o}\}$. *Then the varieties* $\mathbb{FL}^{\ell}_{\mathrm{S}}$, $\mathbb{RL}^{\ell}$ *and* $\mathbb{RL}^{\ell}_{\mathrm{c}}$ *enjoy neither* $\mathbb{Q}\mathrm{C}$ *nor* $\mathbb{R}\mathrm{C}$.

The previous theorem states that some semilinear varieties do not have $\mathbb{Q}\mathrm{C}$ and $\mathbb{R}\mathrm{C}$. Consequently, they do not enjoy also $\mathrm{S}\mathbb{Q}\mathrm{C}$, $\mathrm{FS}\mathbb{Q}\mathrm{C}$, $\mathrm{S}\mathbb{R}\mathrm{C}$ and $\mathrm{FS}\mathbb{R}\mathrm{C}$.

The method by which Theorem 4.2.3 was proved, suggests that a similar approach can be used also for other semilinear varieties. It is sufficient for a given variety to find an algebra where (20) is not valid. One can even use a computer system for searching such counterexamples. The next lemma presents a cyclic involutive $\mathrm{FL_c}$-chain which was found using the computer program *Mace4*.

LEMMA 4.2.4. *There is a cyclic involutive* $\mathrm{FL_c}$*-chain* $\boldsymbol{C}_8$ *where*

$$1 \leq (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x)))$$

does not hold.

Proof. Let $C_8 = \{\bot, a, 1, b, c, 0, d, \top\}$ ordered by $\bot < a < 1 < b < c < 0 < d < \top$ (see Figure 7). The multiplication $\cdot$ is defined as follows:

$\cdot$	$\bot$	a	1	b	c	0	d	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a	c	0	0	$\top$
1	$\bot$	a	1	b	c	0	d	$\top$
b	$\bot$	b	b	b	c	d	d	$\top$
c	$\bot$	c	c	c	$\top$	$\top$	$\top$	$\top$
0	$\bot$	c	0	$\top$	$\top$	$\top$	$\top$	$\top$
d	$\bot$	c	d	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Then the divisions can be computed using Proposition 3.0.9.

We will check that (20) is not valid in $\boldsymbol{C}_8$. Let $x = y = c$ and $z = 0$. Then $z\backslash y = z\backslash x = 0\backslash c = a$ and $x\backslash y = c\backslash c = b$. Consequently,

$$(z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x))) = a \vee (b\backslash(1/a)) = a \vee (b\backslash 1) = a \vee \bot = a < 1\,.$$

Thus $\not\models_{\boldsymbol{C}_8} 1 \leq (z\backslash y) \vee ((x\backslash y)\backslash(1/(z\backslash x)))$. □

It immediately follows from Lemma 4.2.4 that an analogue of Theorem 4.2.3 holds also for cyclic involutive $\mathrm{FL}^{\ell}_{\mathrm{c}}$-algebras and also for semilinear varieties above it.

THEOREM 4.2.5. *Let* $\mathrm{S} = \emptyset$ *or* $\mathrm{S} = \{\mathrm{c}\}$. *Then the varieties* $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$ *and* $\mathbb{C}\mathrm{y}\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$ *enjoy neither* $\mathbb{Q}\mathrm{C}$ *nor* $\mathbb{R}\mathrm{C}$.

Note that the counterexamples given in Lemmas 4.2.2 and 4.2.4 are not commutative. Thus it is natural to ask whether we can obtain some of the completeness properties at least in the commutative case. The answer to this question is affirmative. It was proved in [32] that $\mathbb{FL}^{\ell}_{\mathrm{e}}$ enjoys $\mathrm{S}\mathbb{Q}\mathrm{C}$ and $\mathrm{S}\mathbb{R}\mathrm{C}$. However, the result was proved proof-theoretically using the hypersequent calculus for semilinear FL_{e}. So far there is no algebraic proof of this fact. Thus we present the next theorem without a proof.

THEOREM 4.2.6. *The variety* $\mathbb{FL}^{\ell}_{\mathrm{e}}$ *has the strong completeness property with respect to the class of all countably infinite dense* FL_{e}*-chains and also* FL_{e}*-chains on* $[0, 1]$.

Let us remark that the above theorem was proved not precisely for semilinear FL_{e}-algebras but for semilinear FL_{e}-algebras with an expanded language by constants $\bot$ and $\top$. These constants are interpreted by a bottom and a top element, thus all FL_{e}-algebras in this expanded language are bounded. The reason why we could claim Theorem 4.2.6 follows from Proposition 3.3.1 from which it follows that every FL_{e}-chain can be extended to a bounded FL_{e}-chain.

Concerning the completeness properties for non-integral semilinear varieties with respect to the classes of countably infinite dense chains and chains on $[0, 1]$, there remain still unsolved cases. For instance, the situation is not known for the following semilinear varieties: $\mathbb{FL}^{\ell}_{\mathrm{ec}}$, $\mathbb{RL}^{\ell}_{\mathrm{ec}}$, $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{e}}$ and $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{ec}}$.

4.2.2 Completeness w.r.t. finite algebras

Negative results can be obtained also for the completeness properties with respect to the class of finite chains. Namely, we will show that a bunch of non-integral semilinear varieties of FL-algebras do not enjoy $\mathrm{FS}\mathbb{F}\mathrm{C}$.

Consider the following quasi-identity saying that every positive right-invertible element x has to be equal to 1:

$$1 \leq x \quad \text{and} \quad x \cdot y = 1 \qquad \text{implies} \qquad x = 1\,. \tag{21}$$

This quasi-identity holds in any finite FL-algebra $\boldsymbol{A}$. Indeed, let $x, y \in A$ such that $1 \leq x$ and $xy = 1$. We have $x \leq x^2 \leq x^3 \leq \cdots$ because x is positive. Further, since $\boldsymbol{A}$ is finite, there is $n \in \mathsf{N}$ such that $x^{n+1} = x^n$. Multiplying this equation by y^n from the right-hand side, we obtain $x = 1$. Thus every variety $\mathbb{L}$ of FL-algebras containing a member with a strictly positive invertible element cannot enjoy the finite strong completeness property with respect to the class $\mathbb{F}$ of all finite members of $\mathbb{L}$ because $\{1 \leq x,\ x \cdot y = 1\} \models_{\mathbb{F}} x = 1$ but $\{1 \leq x,\ x \cdot y = 1\} \not\models_{\mathbb{L}} x = 1$. Using this observation, we can prove the following theorem.

THEOREM 4.2.7. *Let* $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{o}\}$. *Then the variety* $\mathbb{FL}^{\ell}_{\mathrm{S}}$ *does not have the finite strong completeness property with respect to the class of all finite* FL_{S}*-chains. The same is true also for* $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$, $\mathbb{C}\mathrm{y}\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$, $\mathbb{RL}^{\ell}_{\mathrm{S}}$, $\mathbb{I}\mathrm{n}\mathbb{RL}^{\ell}_{\mathrm{S}}$ *and* $\mathbb{C}\mathrm{y}\mathbb{I}\mathrm{n}\mathbb{RL}^{\ell}_{\mathrm{S}}$ *if* $\mathrm{S} \subseteq \{\mathrm{e}\}$.

Proof. Consider the additive ℓ-group $\boldsymbol{Z}$ of integers. It forms a cyclic involutive RL_{e}-chain. The involutive RL_{e}-chain $\boldsymbol{Z}$ does not satisfy (21) because every positive integer is invertible. Thus the claim follows for $\mathbb{FL}^{\ell}_{\mathrm{S}}$, $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$, $\mathbb{C}\mathrm{y}\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{S}}$, $\mathbb{RL}^{\ell}_{\mathrm{S}}$, $\mathbb{I}\mathrm{n}\mathbb{RL}^{\ell}_{\mathrm{S}}$ and $\mathbb{C}\mathrm{y}\mathbb{I}\mathrm{n}\mathbb{RL}^{\ell}_{\mathrm{S}}$ if $\mathrm{S} \subseteq \{\mathrm{e}\}$.

If $\{\mathrm{o}\} \subseteq \mathrm{S}$ then we can extend $\boldsymbol{Z}$ to a bounded $\mathrm{FL}_{\mathrm{eo}}$-chain $\boldsymbol{Z}_{\perp}$ by Proposition 3.3.1 interpreting the constant 0 by the bottom element $\perp$. Since $\boldsymbol{Z}_{\perp}$ possesses a strictly positive invertible element, the claim follows for $\mathbb{FL}^{\ell}_{\mathrm{S}}$ also for $\mathrm{S} \subseteq \{\mathrm{e}, \mathrm{o}\}$. $\square$

Thus the above-mentioned varieties are not generated by finite members as quasi-varieties. On the other hand, it is not known whether the above varieties are generated by their finite members, i.e., if they enjoy at least $\mathbb{F}\mathrm{C}$. Also for $\{\mathrm{c}\} \subseteq \mathrm{S} \subseteq \{\mathrm{c}, \mathrm{o}\}$ we do not know whether $\mathbb{FL}^{\ell}_{\mathrm{S}}$ satisfies some of the completeness properties with respect to the class of all finite FL_{S}-chains.

On the other hand, we have at least some positive results. It turns out that contraction together with commutativity is sufficient to prove $\mathrm{FS}\mathbb{F}\mathrm{C}$. It was proved in [40] that the varieties $\mathbb{FL}_{\mathrm{ec}}$ and $\mathbb{FL}_{\mathrm{eco}}$ enjoy $\mathrm{FS}\mathbb{F}\mathrm{C}$. The same was proved also for $\mathbb{I}\mathrm{n}\mathbb{FL}_{\mathrm{ec}}$ and $\mathbb{I}\mathrm{n}\mathbb{FL}_{\mathrm{eco}}$ in [37]. Although it is not mentioned in the above-mentioned papers explicitly, the results easily extend also to the semilinear case.

THEOREM 4.2.8. *The varieties* $\mathbb{FL}^{\ell}_{\mathrm{ec}}$, $\mathbb{FL}^{\ell}_{\mathrm{eco}}$, $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{ec}}$ *and* $\mathbb{I}\mathrm{n}\mathbb{FL}^{\ell}_{\mathrm{eco}}$ *enjoy the finite strong completeness property with respect to the class of all finite chains.*

5 Subvariety lattice

In this section we will study the structure of the subvariety lattice of FL^{ℓ}-algebras. Its structure is very complex and it is not very well known. Even at the very bottom we have continuum many atoms. Thus we focus here only on the cardinalities of atoms in the subvariety lattices of varieties $\mathbb{FL}^{\ell}_{\mathrm{S}}$ for $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i},\mathrm{o}\}$ and $\mathbb{RL}^{\ell}_{\mathrm{S}}$ for $\mathrm{S} \subseteq \{\mathrm{e},\mathrm{c},\mathrm{i}\}$. Note that due to algebraizability the cardinality of atoms in the subvariety lattice of $\mathbb{FL}^{\ell}_{\mathrm{S}}$ (resp. $\mathbb{RL}^{\ell}_{\mathrm{S}}$) corresponds to the cardinality of maximally consistent axiomatic extensions of $\mathrm{FL}^{\ell}_{\mathrm{S}}$ (resp. $\mathrm{RL}^{\ell}_{\mathrm{S}}$).

5.1 General facts

Given a variety $\mathbb{V}$ of FL-algebras, we denote its subvariety lattice $\mathbf{\Lambda}(\mathbb{V})$. Recall Theorem 2.1.1 saying that varieties of algebras are exactly equational classes. Since FL-algebras are defined over a finite language, there are countably many equations over this language. Due to this fact, it follows that $\mathbf{\Lambda}(\mathbb{FL})$ can have cardinality at most $2^{\aleph_0}$.

There are in fact two types of results we are going to prove. The first one are results saying that a subvariety lattice contains continuum many atoms. Such results are proved by constructing $2^{\aleph_0}$ algebras generating pair-wise different atoms in the subvariety lattice. The second type of results shows that there are only finitely many atoms in a subvariety lattice $\mathbf{\Lambda}(\mathbb{L})$. This can be proved by showing that each nontrivial variety $\mathbb{V} \in \mathbf{\Lambda}(\mathbb{L})$ contains an algebra generating one of the finitely many atoms. In both cases we need a criterion whether a given algebra $\boldsymbol{A}$ generates an atom in a subvariety lattice. A summary of the results we are going to prove can be found in Figures 9, 10. The cardinalities of atoms are written in the boxes next to the corresponding semilinear varieties. It is clear that if a variety has a subvariety lattice with $2^{\aleph_0}$ atoms then the same is true also for all varieties above it.

A nontrivial algebra $\boldsymbol{A}$ is said to be *strictly simple* if it lacks nontrivial proper subalgebras and congruences. By *proper* subalgebra of $\boldsymbol{A}$ we mean here a subalgebra $\boldsymbol{B}$ which is not isomorphic to $\boldsymbol{A}$. Note that this notion differs from the usual one saying that a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is proper if $B \subsetneq A$. Strictly simple algebras are good candidates for generators of an atom. Let $\boldsymbol{A}$ be a strictly simple FL-algebra. Since the variety of FL-algebras is congruence distributive, we can use Jónsson's Lemma to show that subdirectly irreducible members in $\mathbf{V}(\boldsymbol{A})$ are contained in $\mathbf{HSP}_{\mathrm{U}}(\boldsymbol{A})$. We will mention two special cases in which strictly simple algebras generate an atom.

First, if $\boldsymbol{A}$ is finite then $\mathbf{P}_{\mathrm{U}}(\boldsymbol{A})$ contains just $\boldsymbol{A}$ because any ultrapower of a finite structure is isomorphic to the original structure. Thus each subdirectly irreducible member has to belong to $\mathbf{HS}(\boldsymbol{A})$ in this case. Since $\boldsymbol{A}$ has no proper nontrivial subalgebras and congruences, we get that $\boldsymbol{A}$ is the only subdirectly irreducible algebra in $\mathbf{V}(\boldsymbol{A})$ which means that $\mathbf{V}(\boldsymbol{A})$ has to be an atom (recall that every variety is generated by its subdirectly irreducible members). Thus we have the following lemma.

LEMMA 5.1.1. *Let $\boldsymbol{A}$ be a finite nontrivial strictly simple* FL*-algebra. Then $\mathbf{V}(\boldsymbol{A})$ is minimal, i.e., it forms an atom.*

Second, we will discuss the case when $\boldsymbol{A}$ is infinite and lower-bounded. Moreover, the lower-bound $\bot$ has to be nearly term definable. An element $b \in A$ is called *nearly term definable* if there is an n-ary term $t(x_1,\dots,x_n)$ such that $t(a_1,\dots,a_n) = b$ holds unless $a_1 = \dots = a_n = 1$.

LEMMA 5.1.2. *Let $\boldsymbol{A}$ be a strictly simple* FL*-algebra with a bottom element $\bot$ nearly term definable by an n-ary term t. Then $\mathbf{V}(\boldsymbol{A})$ is a minimal variety. Moreover, if $\boldsymbol{A}'$ is* *a strictly simple* FL*-algebra with a bottom element nearly term definable by the same term* t*, then* $\mathbf{V}(\boldsymbol{A}) \subseteq \mathbf{V}(\boldsymbol{A}')$ *if and only if* $\boldsymbol{A}$ *and* $\boldsymbol{A}'$ *are isomorphic.*

Proof. Let $\boldsymbol{D}$ be a subdirectly irreducible member of $\mathbf{V}(\boldsymbol{A})$. By Jónsson's Lemma there is an ultrapower $\boldsymbol{B} = \boldsymbol{A}^I/U$, a nontrivial subalgebra $\boldsymbol{C}$ of $\boldsymbol{B}$ and a homomorphism $f\colon C \to D$ such that $f(C) = D$. In order to show that $\mathbf{V}(\boldsymbol{A})$ is minimal, we will show that $\boldsymbol{D}$ contains $\boldsymbol{A}$ as a subalgebra. Clearly $\boldsymbol{B}$ contains an isomorphic copy of $\boldsymbol{A}$ (it is a subalgebra of all congruence classes containing constant functions). We identify the elements from this copy with the original elements from $\boldsymbol{A}$. Thus $\bot \in B$. Observe that $\boldsymbol{A}$ satisfies the following first-order sentence expressing the fact that $t(x_1, \dots, x_n) = \bot$ if at least one of $x_1, \dots, x_n$ is not 1:

$$(\forall x_1, \dots, x_n)(\forall y)\,((x_1 \neq 1 \text{ or } \cdots \text{ or } x_n \neq 1) \quad \text{implies} \quad t(x_1, \dots, x_n) \leq y)\ . \tag{22}$$

Since universal sentences are preserved under taking subalgebras and ultrapowers, $\boldsymbol{C}$ has to satisfy (22) as well. Consequently, $\bot \in C$ because $\bot = t(a_1, \dots, a_n)$ for any $a_1, \dots, a_n \in C$ at least one of them different from 1 (we can choose such elements because $\boldsymbol{C}$ is nontrivial). Moreover, $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{C}$ because it is generated by $\bot$ (recall that $\boldsymbol{A}$ is strictly simple, so it is generated by any element different from 1). Finally, we claim that the restriction of f on A has to be an isomorphism. Indeed, if not then $f(\bot) = 1$ because $\boldsymbol{A}$ is simple (it is a well-known fact from universal algebra that each homomorphic image of a simple algebra is either isomorphic to this algebra or it is trivial). Since $\bot$ is the bottom element of $\boldsymbol{B}$ and $\boldsymbol{C}$ as well, we get $f[C] = \{1\}$ which contradicts $\boldsymbol{D}$ being subdirectly irreducible. Consequently, $\boldsymbol{D}$ contains an isomorphic copy of $\boldsymbol{A}$. Hence $\mathbf{V}(\boldsymbol{A})$ is an atom.

To see the second part of the lemma, assume that $\mathbf{V}(\boldsymbol{A}) \subseteq \mathbf{V}(\boldsymbol{A}')$. Since $\boldsymbol{A}$ is subdirectly irreducible, $\boldsymbol{A}'$ is embeddable into $\boldsymbol{A}$ by the previous argument. As both algebras $\boldsymbol{A}$, $\boldsymbol{A}'$ are generated by $\bot$, the embedding has to be onto. □

Finally, in order to construct algebras generating an atom in a subvariety lattice, we will need a method extending an $\mathrm{RL_{ei}}$-chain by a new neutral element for the multiplication. Let $\boldsymbol{A} = \langle A, \cdot, \to, \wedge, \vee, 1^{\boldsymbol{A}} \rangle$ be an $\mathrm{RL_{ei}}$-chain with a coatom $a = \max(A \setminus \{1^{\boldsymbol{A}}\})$. We will extend the 1-free reduct of $\boldsymbol{A}$ by adding a new neutral element $1^{\boldsymbol{A}^\top}$ in order to obtain an $\mathrm{RL_e}$-chain $\boldsymbol{A}^\top = \langle A^\top, \cdot, \to, \wedge, \vee, 1^{\boldsymbol{A}^\top} \rangle$, where $A^\top = A \cup \{1^{\boldsymbol{A}^\top}\}$. The new lattice order $\wedge, \vee$ is the extension of the original order letting $a \leq 1^{\boldsymbol{A}^\top} \leq 1^{\boldsymbol{A}}$. Thus $1^{\boldsymbol{A}}$ becomes a top element of $\boldsymbol{A}^\top$. Let $x \in A \cup \{1^{\boldsymbol{A}^\top}\}$ and $y \in A \setminus \{1^{\boldsymbol{A}}\}$. The operations are extended as follows:

$$1^{\boldsymbol{A}^\top} \cdot x = x = x \cdot 1^{\boldsymbol{A}^\top}\,, \quad 1^{\boldsymbol{A}^\top} \to x = x\,, \quad y \to 1^{\boldsymbol{A}^\top} = 1^{\boldsymbol{A}}\,, \quad 1^{\boldsymbol{A}} \to 1^{\boldsymbol{A}^\top} = a\,.$$

LEMMA 5.1.3. *Let* $\boldsymbol{A}$ *be an* $\mathrm{RL_{ei}}$*-chain with a coatom* a*. Then* $\boldsymbol{A}^\top$ *is an* $\mathrm{RL_e}$*-chain. Moreover, if* $\boldsymbol{A}$ *is contractive then* $\boldsymbol{A}^\top$ *is contractive as well.*

Proof. It is easy to check that $\langle A^\top, \cdot, 1^{\boldsymbol{A}^\top} \rangle$ forms a commutative monoid. We will check that $\boldsymbol{A}^\top$ satisfies the residuation property, i.e., $xy \leq z$ iff $y \leq x \to z$. Since A is a subuniverse of the 1-free reduct of $\boldsymbol{A}^\top$, we have $x \cdot y \leq z$ iff $y \leq x \to z$ for

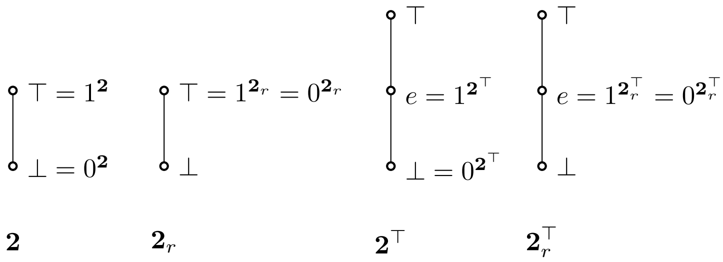

Figure 8. Some strictly simple FL-chains.

all $x, y, z \in A$. Let $x, y, z \in A \cup \{1^{\boldsymbol{A}^\top}\}$ and suppose that at least one of them equals $1^{\boldsymbol{A}^\top}$. If $x = 1^{\boldsymbol{A}^\top}$ then we have $1^{\boldsymbol{A}^\top} \cdot y \leq z$ iff $y \leq z = 1^{\boldsymbol{A}^\top} \to z$. Thus assume $x \neq 1^{\boldsymbol{A}^\top}$. If $z \neq 1^{\boldsymbol{A}^\top}$ (i.e., $y = 1^{\boldsymbol{A}^\top}$) then $x \leq z$ iff $1^{\boldsymbol{A}} \leq x \to z$. Consequently, $x = x \cdot 1^{\boldsymbol{A}^\top} \leq z$ iff $1^{\boldsymbol{A}^\top} \leq x \to z$. Now assume $z = 1^{\boldsymbol{A}^\top}$. If $x = 1^{\boldsymbol{A}}$ then $1^{\boldsymbol{A}} \cdot y \leq 1^{\boldsymbol{A}^\top}$ iff $y \leq a = 1^{\boldsymbol{A}} \to 1^{\boldsymbol{A}^\top}$. If $x \leq a$ then $x \cdot y \leq x \leq 1^{\boldsymbol{A}^\top}$. Thus $x \cdot y \leq 1^{\boldsymbol{A}^\top}$ iff $y \leq 1^{\boldsymbol{A}} = x \to 1^{\boldsymbol{A}^\top}$. Summing up, $\boldsymbol{A}^\top$ is an $\mathrm{RL_e}$-chain. The moreover part follows easily since we have $x \leq x^2$ for $x \in A$ and $1^{\boldsymbol{A}^\top} = (1^{\boldsymbol{A}^\top})^2$. □

Figure 8 shows several examples of strictly simple FL-chains which we will need later. The first one is the two element Boolean algebra which forms an $\mathrm{FL_{cw}}$-chain $\mathbf{2}$. The second one is its corresponding $\mathrm{RL_{ci}}$-chain $\mathbf{2}_r$. The last algebra $\mathbf{2}_r^\top$ is obtained from $\mathbf{2}_r$ by means of Lemma 5.1.3. Thus $\mathbf{2}_r^\top$ is an $\mathrm{RL_{ec}}$-chain (recall that contractivity together with integrality implies commutativity). Finally, the third algebra $\mathbf{2}^\top$ is an $\mathrm{FL_{ec}}$-chain whose 0-free reduct is $\mathbf{2}_r^\top$ and 0 is interpreted as $\bot$. All of them are clearly simple because every nontrivial filter F (i.e., $F \neq \uparrow 1$) has to contain $\bot$. Thus all of them have only two filters. It is also easy to check that they have no proper nontrivial subalgebras. Consequently, each of these chains generates an atom in the subvariety lattice $\mathbf{\Lambda}(\mathbb{FL}^\ell)$ by Lemma 5.1.1.

5.2 Finitely many atoms

We start with the easier part proving that some of the subvariety lattices contain only finitely many atoms. Note that if an FL-algebra $\boldsymbol{A}$ is lower bounded (i.e., it has a bottom element $\bot$), then it is bounded since $\bot\backslash\bot$ is a top element usually denoted $\top$. Moreover the set $\{\bot, \top\}$ is closed under multiplication, left and right divisions, and lattice operations. Thus it forms a subalgebra of the $\{0,1\}$-free reduct of $\boldsymbol{A}$. Consequently, if $\boldsymbol{A}$ is an $\mathrm{FL_w}$-algebra (i.e., $1 = \top$ and $0 = \bot$), then $\{\bot, \top\} = \{0, 1\}$ forms a subalgebra of $\boldsymbol{A}$. Moreover, it is easy to check that this subalgebra is in fact the two element Boolean algebra $\mathbf{2}$. Thus we have the following theorem (note that $\mathbf{2}$ is also commutative and contractive).

THEOREM 5.2.1. *Let* $\{\mathrm{w}\} \subseteq \mathrm{S} \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{w}\}$. *Then* $\mathbf{\Lambda}(\mathbb{FL}_{\mathrm{S}}^\ell)$ *contains a single atom, namely the variety* $\mathbf{V}(\mathbf{2})$ *generated by the two element Boolean algebra* $\mathbf{2}$.

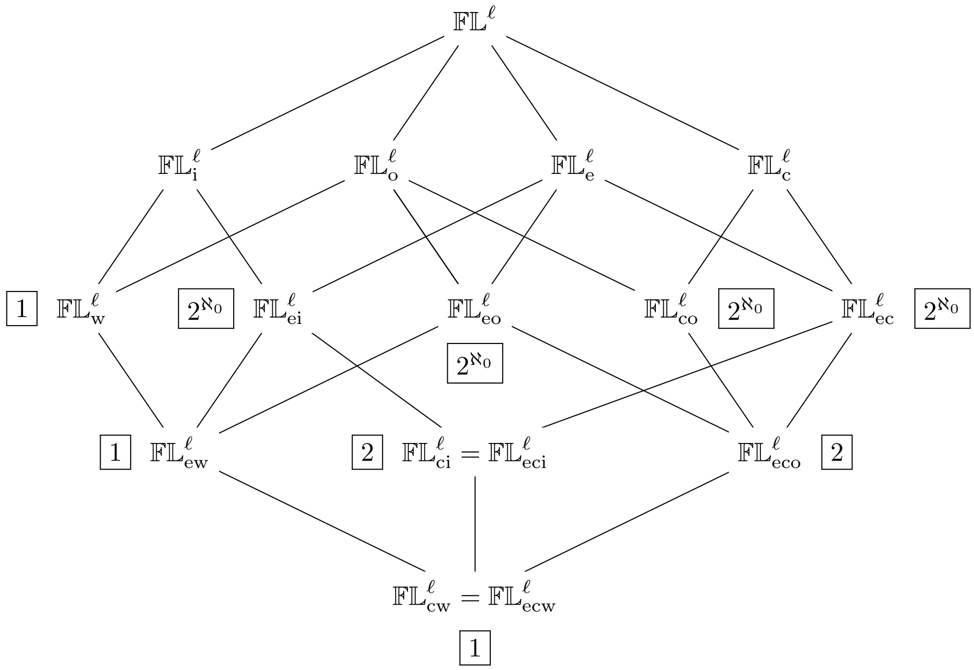

Figure 9. The lattice of considered semilinear varieties of $\mathrm{FL_S}$-algebras and the numbers of atoms in their corresponding subvariety lattices.

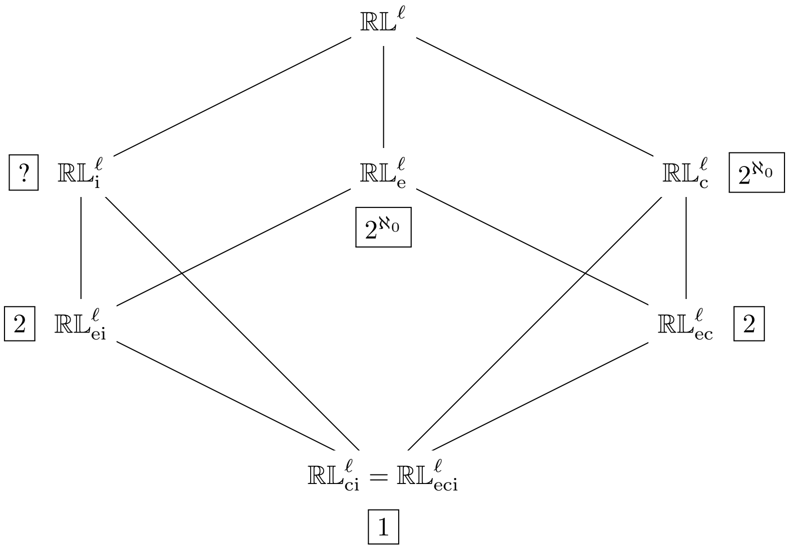

Figure 10. The lattice of considered semilinear varieties of $\mathrm{RL_S}$-algebras and the numbers of atoms in their corresponding subvariety lattices.

The above theorem covers all cases where $\{\mathrm{w}\} \subseteq \mathrm{S}$. Thus in the following text we have to omit either the condition that 1 is a top element or that 0 is a bottom element. If we remove the second condition, we obtain the following theorem.

THEOREM 5.2.2. *The subvariety lattice $\mathbf{\Lambda}(\mathbb{FL}^\ell_{\mathrm{ci}})$ contains two atoms, namely the varieties $\mathbf{V}(\mathbf{2})$ and $\mathbf{V}(\mathbf{2}_r)$.*

Proof. Let $\boldsymbol{A}$ be a nontrivial algebra from $\mathbb{FL}^\ell_{\mathrm{ci}}$. Since every algebra in $\mathbb{FL}^\ell_{\mathrm{ci}}$ is integral, 1 is the top element. Moreover, $xy = x \wedge y$ because $\boldsymbol{A}$ is contractive. If $0 < 1$ then $\{0, 1\}$ forms a subalgebra of $\boldsymbol{A}$ isomorphic to $\mathbf{2}$. Thus assume $1 = 0$. Let $a < 1$ be a strictly negative element from A (see Lemma 3.0.14). Then $\{a, 1\}$ forms a subalgebra of $\boldsymbol{A}$ isomorphic to $\mathbf{2}_r$. $\square$

If we remove integrality from $\mathbb{FL}^\ell_{\mathrm{cw}}$, the number of atoms also increases as in the previous theorem.

THEOREM 5.2.3. *The subvariety lattice $\mathbf{\Lambda}(\mathbb{FL}^\ell_{\mathrm{eco}})$ contains two atoms, namely varieties $\mathbf{V}(\mathbf{2})$ and $\mathbf{V}(\mathbf{2}^\top)$.*

Proof. Let $\boldsymbol{A}$ be a subdirectly irreducible algebra from $\mathbb{FL}^\ell_{\mathrm{eco}}$ (i.e., $\boldsymbol{A}$ is a nontrivial chain). Then 0 is the bottom element. Consider the $\mathrm{FL_{eco}}$-chain $\boldsymbol{B} = \boldsymbol{A}/F$, where F is the maximal filter not containing 0 (i.e., the union of all filters not containing 0; it is a filter because $\boldsymbol{A}$ is a chain). The algebra $\boldsymbol{B}$ is clearly simple. Recall that $xy = x \wedge y$ for negative elements $x, y \in B$ because $\boldsymbol{B}$ is contractive. We claim that 0 is a subcover of 1. Indeed, if there would be an element $b \in B$ such that $0 < b < 1$ then the filter

$F(b) = \uparrow b$ generated by b is nontrivial and proper (see Theorem 3.5.2). Thus $\boldsymbol{B}$ would not be simple. Further $\boldsymbol{B}$ has a top element $\top = 0 \to 0$. If $\top = 1$ then $\{0, 1\}$ forms a subalgebra of $\boldsymbol{B}$ isomorphic to $\mathbf{2}$. If $\top > 1$ then we claim that $\{0, 1, \top\}$ is a subuniverse of $\boldsymbol{B}$ isomorphic to $\mathbf{2}^\top$. The set $\{0, 1, \top\}$ is clearly closed under multiplication. Recall that $\{0, \top\}$ is closed under $\to$. Furthermore we have $1 \to x = x$ and $0 \to 1 = \top$. Thus the only remaining case is $\top \to 1$. The value $\top \to 1$ has to be strictly less than 1. Since 0 is the subcover of 1, we must have $\top \to 1 = 0$. It is straightforward to check that the subuniverse $\{0, 1, \top\}$ forms an algebra isomorphic to $\mathbf{2}^\top$. $\square$

Now we turn our attention to RL^ℓ-algebras. The situation for $\mathrm{RL}^\ell_{\mathrm{ci}}$-algebras is again easy. Let $\boldsymbol{A}$ be a nontrivial algebra from $\mathbf{\Lambda}(\mathbb{RL}^\ell_{\mathrm{ci}})$ and $a \in A$ a strictly negative element (see Lemma 3.0.14). Since the multiplication coincides with $\wedge$, the set $\{a, 1\}$ forms a subalgebra of $\boldsymbol{A}$ isomorphic to $\mathbf{2}_r$. Thus we obtain the following theorem.

THEOREM 5.2.4. *The subvariety lattice* $\mathbf{\Lambda}(\mathbb{RL}^\ell_{\mathrm{ci}})$ *contains a single atom, namely the variety* $\mathbf{V}(\mathbf{2}_r)$.

We have seen that the subvariety lattice of contractive integral RL-chains contains only a single atom. Even without integrality the situation is relatively easy if we have commutativity.

THEOREM 5.2.5. *The subvariety lattice* $\mathbf{\Lambda}(\mathbb{RL}^\ell_{\mathrm{ec}})$ *contains two atoms, namely varieties* $\mathbf{V}(\mathbf{2}_r)$ *and* $\mathbf{V}(\mathbf{2}_r^\top)$.

Proof. Let $\boldsymbol{A}$ be a subdirectly irreducible member of $\mathbf{\Lambda}(\mathbb{RL}^\ell_{\mathrm{ec}})$. Recall that $xy = x \wedge y$ for $x, y \leq 1$ because $\boldsymbol{A}$ is contractive. Thus every negative element $a \in A$ generates a filter $F(a) = \uparrow a$ (see Theorem 3.5.2). Since $\boldsymbol{A}$ is subdirectly irreducible, there has to be minimum nontrivial filter $F(a)$ for a strictly negative $a \in A$ (see Theorems 2.1.4 and 3.5.3). Then a is a subcover of 1. Set $T = a \to 1$. We claim that $\{a, 1, T\}$ is a subalgebra of $\boldsymbol{A}$. If $T = 1$ then $\{a, 1\}$ is clearly closed under multiplication. It is also closed under $\to$ because $1 \to x = x$ for every $x \in A$ and

$$1 \leq a \to a \leq a \to 1 = T = 1\,.$$

Thus $\{a, 1\}$ forms a subalgebra isomorphic to $\mathbf{2}_r$.

Thus assume that $T > 1$. We check that $\{a, 1, T\}$ is closed under multiplication. Clearly, $1x = x = x1$ for $x \in \{a, 1, T\}$. Further, $a^2 = a$. To see that $aT = Ta \in \{a, 1, T\}$, note that

$$a \leq aT = a(a \to 1) \leq 1\,.$$

Since a is a subcover of 1, we get $Ta = aT \in \{a, 1\}$. Moreover, $Ta \neq 1$. Indeed, if $Ta = 1$ then $a = (Ta)a = Ta^2 = Ta = 1$. Thus $Ta = aT = a$. Consequently, $aT^2 = (aT)T = a \leq 1$. Thus $T \leq T^2 \leq a \to 1 = T$. Hence $\{a, 1, T\}$ is closed under multiplication. Finally, we have to check that $\{a, 1, T\}$ is closed under $\to$. Let $x \in \{a, 1, T\}$. Clearly, $1 \to x = x$. Next we show that $a \to x \in \{a, 1, T\}$. We have $a \to 1 = T$ by the definition. Then $a \to T = a \to (a \to 1) = a^2 \to 1 = a \to 1 = T$. Also $aT = a$, hence

$$T \leq a \to a \leq a \to 1 = T\,.$$

Finally, we show that $T \to x \in \{a, 1, T\}$. We have $T \to 1 = (a \to 1) \to 1 \geq a$. On the other hand, $T \to 1 < 1$ otherwise $1 \geq T(T \to 1) \geq T$. Thus $T \to 1 = a$ because a is the subcover of 1. Moreover, $a \leq T \to a$ because $Ta = a$. On the other hand, $T \to a < 1$ otherwise $a \geq T(T \to a) \geq T$. Thus $T \to a = a$. Finally, to see that $T \to T \in \{a, 1, T\}$, note that $T^2 = T$ and $aT = a$. Consequently,

$$T \leq T \to T = T \to (a \to 1) = aT \to 1 = a \to 1 = T\,.$$

Thus $\{a, 1, T\}$ forms a subalgebra of $\boldsymbol{A}$. It is easy to see that it is isomorphic to $\mathbf{2}_r^{\top}$. □

If we replace contraction by integrality, the number of atoms still remains finite. Recall that $\boldsymbol{Z} = \langle \mathsf{Z}, +, \to, \wedge, \vee, 0\rangle$ (where $x \to y = y - x$) denotes the additive ℓ-group of integers viewed as an $\mathrm{RL_e}$-chain. One can apply to $\boldsymbol{Z}$ the conucleus $\sigma(x) = x \wedge 0$ (see Section 3.7) in order to obtain an $\mathrm{RL_{ei}}$-chain $\boldsymbol{Z}^- = \sigma[\boldsymbol{Z}] = \langle \mathsf{Z}^-, +, \to_\sigma, \wedge, \vee, 0\rangle$. Note that $x \to_\sigma y = (y - x) \wedge 0$. Before we prove that the subvariety lattice $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{ei}})$ contains only two atoms, we will prove that $\boldsymbol{Z}^-$ generates an atom.

LEMMA 5.2.6. *The variety* $\mathbf{V}(\boldsymbol{Z}^-)$ *is an atom in* $\mathbf{\Lambda}(\mathbb{RL}_{\mathrm{ei}})$.

Proof. Assume that $\mathbf{V}(\boldsymbol{A}) \subseteq \mathbf{V}(\boldsymbol{Z}^-)$ is a nontrivial subvariety of $\mathbf{V}(\boldsymbol{Z}^-)$. Thus $\boldsymbol{A}$ is a nontrivial $\mathrm{RL_{ei}}$-algebra. First, observe that $\boldsymbol{Z}^-$ satisfies the identity $x \to xy = y$. Indeed, we have $x \to_\sigma (x + y) = (x + y - x) \wedge 0 = y$. Thus $\mathbf{V}(\boldsymbol{A})$ has to satisfy this identity as well. Let a be a strictly negative element in A (see Lemma 3.0.14). Consider the submonoid B of $\boldsymbol{A}$ generated by a, i.e., $B = \{a^k \mid k \in \mathsf{N}\}$. We claim that $a^{k+1} < a^k$ for all $k \in \mathsf{N}$. Indeed, if $a^{k+1} = a^k$ for some $k \in \mathsf{N}$, then

$$1 = a^k \to a^k = a^k \to a^{k+1} = a\,.$$

Further, we claim that B forms a subalgebra of $\boldsymbol{A}$. It is clearly closed under $\wedge, \vee$ because B is a chain. It remains to check that B is closed under $\to$. Let $k, n \in \mathsf{N}$. If $k \leq n$ then $a^n \leq a^k$. Thus $a^n \to a^k = 1 \in B$. Suppose that $k > n$. Then $a^n \to a^k = a^n \to a^n a^{k-n} = a^{k-n}$. Thus B forms a subalgebra which is obviously isomorphic to $\boldsymbol{Z}^-$. Consequently, $\mathbf{V}(\boldsymbol{Z}^-) \subseteq \mathbf{V}(\boldsymbol{A})$, i.e., $\mathbf{V}(\boldsymbol{Z}^-)$ is an atom. □

THEOREM 5.2.7. *The subvariety lattice* $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{ei}})$ *contains two atoms, namely varieties* $\mathbf{V}(\mathbf{2})$ *and* $\mathbf{V}(\boldsymbol{Z}^-)$.

Proof. Let $\boldsymbol{A}$ be a subdirectly irreducible $\mathrm{RL}^{\ell}_{\mathrm{ei}}$-algebra, i.e., $\boldsymbol{A}$ is nontrivial and linearly ordered. Thus $\boldsymbol{A}$ contains is a strictly negative element $a < 1$ as $\boldsymbol{A}$ is integral. We will show that $\mathbf{V}(\boldsymbol{A})$ contains either $\mathbf{2}$ or $\boldsymbol{Z}^-$. Let N^+ denote the set of strictly positive natural numbers and consider the non-increasing sequence $\langle a^k\rangle_{k \in \mathsf{N}^+}$. If $a^{k+1} = a^k$ for some $k \in \mathsf{N}^+$, then a^k is idempotent. Indeed, we have

$$a^{2k} = a^{k+1}a^{k-1} = a^k a^{k-1} = \cdots = a^k\,.$$

Then it is easy to see that $\{a^k, 1\}$ forms a subalgebra of $\boldsymbol{A}$ isomorphic to $\mathbf{2}_r$. Thus in this case we have $\mathbf{2}_r \in \mathbf{V}(\boldsymbol{A})$.

Now assume that $a^{k+1} < a^k$ holds for all $k \in \mathsf{N}^+$, i.e., $\langle a^k \rangle_{k \in \mathsf{N}^+}$ is a strictly decreasing sequence. We will show that $\boldsymbol{Z}^- \in \mathbf{ISHP}_{\mathrm{U}}(\boldsymbol{A}) \subseteq \mathbf{V}(\boldsymbol{A})$. Consider a non-principal ultrafilter U on N and the corresponding ultrapower $\boldsymbol{B} = \boldsymbol{A}^{\mathsf{N}}/U$. Set $\vec{1} = \langle 1, 1 \ldots \rangle/U$, $\vec{a} = \langle a, a, \ldots \rangle/U$, and $\vec{b} = \langle a^k \rangle_{k \in \mathsf{N}^+}/U$. Let θ be the congruence on $\boldsymbol{B}$ corresponding to the filter $F(\vec{a})$ generated by $\vec{a}$ (cf. Theorem 3.5.3). In particular, we have $\vec{1}/\theta = \vec{a}/\theta$ because $\vec{1} \to \vec{a} = \vec{a} \in F(\vec{a})$ and $\vec{a} \to \vec{1} = \vec{1} \in F(\vec{a})$. Further, note that $\vec{b} \notin F(\vec{a})$. Indeed, if $\vec{b} \in F(\vec{a}) = \uparrow\{\vec{a}^n \mid n \in \mathsf{N}\}$, then for a fixed $n \in \mathsf{N}$ and a set J in the ultrafilter U we have $a^k \geq a^n$ for all $k \in J$. However, $\langle a^k \rangle_{k \in \mathsf{N}^+}$ is strictly decreasing. Thus J has to be finite. Consequently, U has to be principal (a contradiction).

We claim that the subalgebra of $\boldsymbol{B}/\theta$ generated by $\vec{b}/\theta$ is isomorphic to $\boldsymbol{Z}^-$. First, it is not trivial because $\vec{b} \notin F(\vec{a})$. Second, we have to check that $\vec{b}^m/\theta \to \vec{b}^n/\theta = \vec{b}^{n-m}/\theta$ for $n > m$. Note that for all $k \in \mathsf{N}^+$ we have

$$a^{(n-m)k} \leq a^{mk} \to a^{nk} < a^{(n-m)k-1} \leq a \to a^{(n-m)k}\,.$$

Thus $\vec{b}^{n-m}/\theta \leq \vec{b}^m/\theta \to \vec{b}^n/\theta \leq \vec{a}/\theta \to \vec{b}^{n-m}/\theta = \vec{1}/\theta \to \vec{b}^{n-m}/\theta = \vec{b}^{n-m}/\theta$. □

Unfortunately, the proof of Theorem 5.2.7 does not work without commutativity (i.e., for the variety $\mathbb{RL}^{\ell}_{\mathrm{i}}$) because the filter $F(\vec{a})$ is closed also under conjugates. Then it is not clear how to prove that $\vec{b} \notin F(\vec{a})$.

PROBLEM 5.2.8. *What is the cardinality of atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{i}})$?*

5.3 Continuum many atoms

Now we focus on varieties whose subvariety lattice has continuum many atoms. We start with the variety of $\mathrm{RL}^{\ell}_{\mathrm{e}}$-algebras and construct $2^{\aleph_0}$ algebras generating atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{e}})$. Easy modifications of these algebras give also $2^{\aleph_0}$ atoms in $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{eo}})$ and $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{ei}})$.

Let $\boldsymbol{A} = \langle A, +, \to, \wedge, \vee, \langle 0, 0 \rangle \rangle$ be the totally ordered Abelian ℓ-group (viewed as an RL_{e}-chain) given by the lexicographic product of two copies of $\boldsymbol{Z}$, i.e., $A = \mathsf{Z}^2$ ordered lexicographically, $+$ is computed component-wise and $\langle x, y \rangle \to \langle u, v \rangle = \langle u - x, v - y \rangle$. Recall that there are $2^{\aleph_0}$ infinite subsets of $-2 + \mathsf{Z}^- = \{-2 + z \mid z \in \mathsf{Z}^-\}$, where Z^- denotes the set of non-positive integers. We construct for each infinite subset $S \subseteq -2 + \mathsf{Z}^-$ an algebra $\boldsymbol{A}_S$ from the algebra $\boldsymbol{A}$ by means of a conucleus σ_S and a nucleus γ. We define the conucleus σ_S by its image as follows:

$$\sigma_S[A] = \{\langle 0, 0 \rangle, \langle -1, 0 \rangle, \langle -1, -1 \rangle\} \cup \{\langle -1, z \rangle \in A \mid z \in S\} \cup \{\langle x, y \rangle \in A \mid x \leq -2\}\,.$$

To show that $\sigma_S[A]$ defines a conucleus σ_S, we have to prove that $\sigma_S[A]$ is the image of an interior operator which forms a submonoid (see Lemma 3.7.1). The set $\sigma_S[A]$ clearly forms a submonoid of $\boldsymbol{A}$ since $\langle -1, x \rangle + \langle -1, y \rangle = \langle -2, x + y \rangle \in \sigma_S[A]$. Further we have to check that $\sigma_S[A]$ is an interior system of $\boldsymbol{A}$, i.e., if $\max\{x \in \sigma_S[A] \mid x \leq y\}$ exists for every $y \in A$. The existence is obvious for $y \in \sigma_S[A]$. Suppose that $y \notin \sigma_S[A]$. Then y has the first component greater than or equal to -1. If $y = \langle u, v \rangle$ for $u > 0$, then the maximum is $\langle 0, 0 \rangle$. If $u = 0$ then the maximum is $\langle -1, 0 \rangle$. If $u = -1$ then the

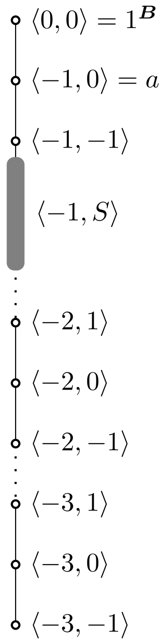

Figure 11. The structure of the algebra $\boldsymbol{B}$.

maximum exists since the set $\langle -1,S\rangle = \{\langle -1,z\rangle \in A \mid z \in S\}$ is dually well-ordered and infinite (thus there is a lower bound of y in $\langle -1,S\rangle$). Hence $\sigma_S[A]$ is the image of a conucleus and $\sigma_S[\boldsymbol{A}] = \langle \sigma_S[A], +, \to_{\sigma_S}, \wedge, \vee, \langle 0,0\rangle\rangle$ forms an $\mathrm{RL}_{\mathrm{ei}}$-chain, where $\langle x,y\rangle \to_{\sigma_S} \langle u,v\rangle = \sigma_S(u-x, v-y)$.

Now consider the γ-retraction $\boldsymbol{B} = \gamma[\sigma_S[\boldsymbol{A}]]$ where $\gamma(x,y) = \langle x,y\rangle \vee \langle -3,-1\rangle$. The algebra $\boldsymbol{B} = \langle \gamma[\sigma_S[A]], +_\gamma, \to_{\sigma_S}, \wedge, \vee, \langle 0,0\rangle\rangle$ is again an $\mathrm{RL}_{\mathrm{ei}}$-chain as follows from Lemma 3.7.4, where $\langle x,y\rangle +_\gamma \langle u,v\rangle = \gamma(x+u, y+v)$. The structure of $\boldsymbol{B}$ is depicted in Figure 11. We define the algebra $\boldsymbol{A}_S$ as the subalgebra of $\boldsymbol{B}$ generated by $a = \langle -1,0\rangle$. The algebra $\boldsymbol{A}_S$ is an $\mathrm{RL}_{\mathrm{ei}}$-chain. It is even a 4-potent $\mathrm{RL}_{\mathrm{ei}}$-chain (i.e., it satisfies $x^5 = x^4$) because $a^4 = \gamma(-4,0) = \langle -3,1\rangle$ is the bottom element.

We will prove that each $\boldsymbol{A}_S$ contains some important elements from B. This will help us to prove that $\boldsymbol{A}_R$ and $\boldsymbol{A}_S$ are not isomorphic for $R \neq S$. First, note that there is a term $r(x) = x^2 \to x^4$ such that its value in $\boldsymbol{A}_S$ for $x = a$ is

$$r^{\boldsymbol{A}_S}(a) = \langle -2,0\rangle \to_{\sigma_S} \langle -3,-1\rangle = \sigma_S(-1,-1) = \langle -1,-1\rangle\,.$$

Thus the element $\langle -1,-1\rangle$ belongs to A_S. Since the set S is dually well ordered, we can index its elements by natural numbers, i.e., $S = \{c_0 > c_1 > c_2 > \cdots\}$.

LEMMA 5.3.1. *For every $\boldsymbol{A}_S$ and every $n \in \mathsf{N}$ there is a term $s_n(x)$ such that*

$$s_n^{\boldsymbol{A}_S}(a) = \langle -1, c_n\rangle\,.$$

Thus $\langle -1, c_n \rangle \in A_S$ *for all* $n \in \mathsf{N}$.

Proof. By induction on n. Let $s_0(x)$ be the term $x \to r(x)^2$. Then

$$s_0(a) = \langle -1, 0\rangle \to_{\sigma_S} \langle -2, -2\rangle = \sigma_S(-1, -2) = \langle -1, c_0\rangle\,,$$

since $\langle -1, c_0\rangle$ is the subcover of $\langle -1, -1\rangle$. Now assume that there is a term $s_n(x)$ such that $s_n^{\boldsymbol{A}_S}(a) = \langle -1, c_n\rangle$. Let $s_{n+1}(x) = x \to r(x) \cdot s_n(x)$. Then

$$s_{n+1}^{\boldsymbol{A}_S}(a) = \langle -1, 0\rangle \to_{\sigma_S} \langle -2, c_n - 1\rangle = \sigma_S(-1, c_n - 1) = \langle -1, c_{n+1}\rangle\,.$$

□

LEMMA 5.3.2. *For every* $\boldsymbol{A}_S$ *and every* $n \in \mathsf{N}$ *there is a term* $t_n(x)$ *such that*

$$t_n^{\boldsymbol{A}_S}(a) = \langle -2, n\rangle\,.$$

Thus $\langle -2, n\rangle \in A_S$ *for all* $n \in \mathsf{N}$.

Proof. By induction on n. For $n = 0$ it is clear since $\langle -2, 0\rangle = a +_\gamma a = a + a$. Assume that there is $t_n(x)$ such that $t_n^{\boldsymbol{A}_S}(a) = \langle -2, n\rangle$. Consider the term $t_{n+1}(x) = r(x) \to x \cdot t_n(x)$. Then

$$t_{n+1}^{\boldsymbol{A}_S}(a) = \langle -1, -1\rangle \to_{\sigma_S} \langle -3, n\rangle = \sigma_S(-2, n+1) = \langle -2, n+1\rangle\,.$$

□

The algebra $\boldsymbol{A}_S$ is a simple $\mathrm{RL}^{\ell}_{\mathrm{ei}}$-algebra because every nontrivial filter F contains a and a^4 is the bottom element, i.e., $F = A_S$. However, $\boldsymbol{A}_S$ is not strictly simple since $\{\langle -3, -1\rangle, \langle 0, 0\rangle\}$ forms a subalgebra isomorphic to $\mathbf{2}_r$. In order to obtain a strictly simple algebra, we will use the construction from Lemma 5.1.3 and extend $\boldsymbol{A}_S$ by a new neutral element.

LEMMA 5.3.3. *The algebra* $\boldsymbol{A}_S^\top$ *is a strictly simple* $\mathrm{RL}^{\ell}_{\mathrm{e}}$*-algebra with a nearly term definable bottom element.*

Proof. We will show that any element $x \neq 1^{\boldsymbol{A}_S^\top}$ generates $\boldsymbol{A}_S^\top$. First, we can make the top element $\langle 0, 0\rangle = x \to x$. Second, we can produce $a = \langle -1, 0\rangle = \langle 0, 0\rangle \to 1^{\boldsymbol{A}_S^\top}$ since $\langle -1, 0\rangle$ is the coatom of $\boldsymbol{A}_S$. Since $\boldsymbol{A}_S$ is generated by a, we are done. Moreover the bottom element a^4 is nearly term definable by the term $x^4 \wedge (x \to 1)^4$. □

In order to invoke Lemma 5.1.2, we have to prove that $\boldsymbol{A}_R$ and $\boldsymbol{A}_S$ are not isomorphic for different sets R, S.

LEMMA 5.3.4. *Let* $R, S \subseteq -2 + \mathsf{Z}^-$ *such that* $R \neq S$. *Then* $\boldsymbol{A}_S$ *is not isomorphic to* $\boldsymbol{A}_R$. *The same is true also for* $\boldsymbol{A}_S^\top$ *and* $\boldsymbol{A}_R^\top$.

Proof. Let us enumerate the elements of R, S as follows: $R = \{d_0 > d_1 > d_2 > \cdots\}$ and $S = \{c_0 > c_1 > c_2 > \cdots\}$. Suppose that $f\colon A_S \to A_R$ is an isomorphism. Since f is order-preserving, f must be the identity when restricted to the set

$$\{\langle 0,0\rangle, \langle -1,0\rangle, \langle -1,-1\rangle, \langle -3,0\rangle, \langle -3,-1\rangle\}\,.$$

By Lemma 5.3.1 we have $\langle -1, c_n\rangle \in A_S$, $\langle -1, d_n\rangle \in A_R$ and $f(-1,c_n) = f(s_n^{\boldsymbol{A}_S}(a)) = s_n^{\boldsymbol{A}_R}(f(a)) = s_n^{\boldsymbol{A}_R}(a) = \langle -1, d_n\rangle$ for all $n \in \mathsf{N}$. Assume that k is the least natural number such that $c_k \neq d_k$. Without any loss of generality suppose that $c_k > d_k$. Lemma 5.3.2 has two consequences. First, $\langle -2, -c_k\rangle \in A_S, A_R$. Second, we have $f(-2,n) = f(t_n^{\boldsymbol{A}_S}(a)) = t_n^{\boldsymbol{A}_R}(f(a)) = t_n^{\boldsymbol{A}_R}(a) = \langle -2, n\rangle$ for all $n \in \mathsf{N}$. Thus we get

$$\begin{aligned} a^3 = f(a^3) = f(-3,0) = f(\langle -1,c_k\rangle + \langle -2,-c_k\rangle) = f(\langle -1,c_k\rangle +_\gamma \langle -2,-c_k\rangle) = \\ \langle -1,d_k\rangle +_\gamma \langle -2,-c_k\rangle = \langle -3, d_k - c_k\rangle \vee \langle -3,-1\rangle = \langle -3,-1\rangle = a^4\,, \end{aligned}$$

which is a contradiction since $a^3 \neq a^4$.

It is straightforward to check that the above argument works also for extended algebras $\boldsymbol{A}_S^\top$ and $\boldsymbol{A}_R^\top$. □

Now using Lemma 5.1.2, we know that each $\boldsymbol{A}_S^\top$ generates an atom in $\boldsymbol{\Lambda}(\mathbb{RL}_{\mathrm{e}}^\ell)$. Moreover, distinct subsets R, S of $-2 + \mathsf{Z}^-$ generate different atoms because $\boldsymbol{A}_R^\top$ and $\boldsymbol{A}_S^\top$ are not isomorphic. Thus we obtain the following theorem.

THEOREM 5.3.5. *There are* $2^{\aleph_0}$ *atoms in* $\boldsymbol{\Lambda}(\mathbb{RL}_{\mathrm{e}}^\ell)$.

The above-mentioned $\mathrm{RL_{ei}}$-chains $\boldsymbol{A}_S$ can be also used to construct atoms in the subvariety lattices $\boldsymbol{\Lambda}(\mathbb{FL}_{\mathrm{eo}}^\ell)$ and $\boldsymbol{\Lambda}(\mathbb{FL}_{\mathrm{ei}}^\ell)$. First, $\boldsymbol{A}_S^\top$ can be easily viewed as an $\mathrm{FL_{eo}}$-chain if we interpret the constant 0 as a^4. Then the premises of Lemma 5.1.2 remain still satisfied. Thus $\mathrm{FL_{eo}}$-chains $\boldsymbol{A}_S^\top$ generate $2^{\aleph_0}$ atoms in $\boldsymbol{\Lambda}(\mathbb{FL}_{\mathrm{eo}}^\ell)$. Second, observe that we can make $\boldsymbol{A}_S$ into an $\mathrm{FL_{ei}}$-chain by interpreting the constant 0 as $a = \langle -1, 0\rangle$. Consequently, the $\mathrm{FL_{ei}}$-chain $\boldsymbol{A}_S$ becomes strictly simple because it is generated by a. Moreover, the bottom element of $\boldsymbol{A}_S$ is term-definable by the term 0^4. Consequently, using Lemma 5.1.2 again, we obtain the next theorem.

THEOREM 5.3.6. *There are* $2^{\aleph_0}$ *atoms in* $\boldsymbol{\Lambda}(\mathbb{FL}_{\mathrm{ei}}^\ell)$ *and* $\boldsymbol{\Lambda}(\mathbb{FL}_{\mathrm{eo}}^\ell)$.

We have seen that the subvariety lattices of $\mathrm{FL}_{\mathrm{ei}}^\ell$-algebras and $\mathrm{FL}_{\mathrm{eo}}^\ell$-algebras have continuum many atoms. The same can be proved also for the subvariety lattice of $\mathrm{FL}_{\mathrm{ec}}^\ell$-algebras. Let $\boldsymbol{A} = \langle \mathsf{Z}^2 \cup \{\bot, \top\}, \cdot, \to, \wedge, \vee, \langle 0,0\rangle\rangle$ be the $\mathrm{RL_e}$-chain arising on the lexicographic product of two copies of the additive ℓ-group of integers Z extended by bottom and top elements $\bot, \top$ (see Proposition 3.3.1). Thus we have $\bot \cdot x = \bot$, $\top \cdot x = \top$ for $x \neq \bot$, and $\langle a_1, b_1\rangle \cdot \langle a_2, b_2\rangle = \langle a_1 + a_2, b_1 + b_2\rangle$. The division $\to$ is computed by formulas $\bot \to x = \top$, $x \to \bot = \bot$ for $x \neq \bot$, $x \to \top = \top$, $\top \to x = \bot$ for $x \neq \top$, and $\langle a_1, b_1\rangle \to \langle a_2, b_2\rangle = \langle a_2 - a_1, b_2 - b_1\rangle$. We define a conucleus σ on $\boldsymbol{A}$ by its image as follows:

$$\sigma[A] = \{\bot, \top, \langle 0,0\rangle\} \cup \{\langle a,b\rangle \mid a > 0, b < 0\}\,.$$

To prove that $\sigma[A]$ is the image of a conucleus, we have to check that $\sigma[A]$ is a submonoid and the image of an interior operator (see Lemma 3.7.1). It is easy to see that $\sigma[A]$ is closed under multiplication, i.e., it forms a submonoid. Further, we have to check that for every $y \in \mathsf{Z}^2 \cup \{\bot, \top\}$ the maximum $m = \max\{x \in \sigma[A] \mid x \leq y\}$ exists. For $y \in \sigma[A]$ it is obvious. Thus suppose that $y \notin \sigma[A]$. If $y < \langle 0,0\rangle$ then the maximum $m = \bot$ because it is the only strictly negative element in $\sigma[A]$. If $y > \langle 0,0\rangle$ then $y = \langle a,b\rangle$ for $a \geq 0$ and $b \geq 0$. Consequently, $m = \langle 0,0\rangle$ if $a = 0$ and $m = \langle a,-1\rangle$ if $a > 0$. Thus $\sigma[A]$ is the image of a conucleus σ and the σ-contraction $\sigma[\boldsymbol{A}] = \langle \sigma[A], \cdot, \rightarrow_\sigma, \wedge, \vee, \langle 0,0\rangle\rangle$ is an $\mathrm{RL_e}$-chain.

Let N^+ denote the set of strictly positive natural numbers. Given a subset $S \subseteq \mathsf{N}^+$ such that $1 \in S$, we define a nucleus γ_S as follows:

$$\gamma_S(x) = x \text{ for } x \in \{\bot, \top, \langle 0,0\rangle\},$$

$$\gamma_S(a,b) = \begin{cases} \langle a,-1\rangle & \text{if } a \notin S \text{ or } b = -1, \\ \langle a,-2\rangle & \text{otherwise.} \end{cases}$$

Observe that the image of γ_S is the following set:

$$\gamma_S[\sigma[A]] = \{\bot, \top, \langle 0,0\rangle\} \cup \{\langle k,-1\rangle \mid k \in \mathsf{N}^+\} \cup \{\langle k,-2\rangle \mid k \in S\}\,.$$

The image of γ_S is well-ordered and contains $\top$. Thus γ_S is a closure operator. We have to check that $\gamma_S(x)\gamma_S(y) \leq \gamma_S(xy)$. If x or y equals $\bot$, $\top$ or $\langle 0,0\rangle$ then the inequality obviously holds. Thus assume that $x, y \notin \{\bot, \top, \langle 0,0\rangle\}$. Then $x = \langle a_1, b_1\rangle$ and $y = \langle a_2, b_2\rangle$ for some $a_1, a_2 \geq 1$ and $b_1, b_2 \leq -1$. We have $\gamma_S(x)\gamma_S(y) \leq \langle a_1 + a_2, -2\rangle$ and $\gamma_S(xy) \geq \langle a_1 + a_2, -2\rangle$. Thus $\gamma_S(x)\gamma_S(y) \leq \gamma_S(xy)$ holds.

Using the conucleus σ and nucleus γ_S, we define an $\mathrm{FL_e}$-chain

$$\boldsymbol{A}_S = \gamma_S[\sigma[\boldsymbol{A}]] = \langle \gamma_S[\sigma[A]], \circ_{\gamma_S}, \rightarrow_\sigma, \wedge, \vee, \langle 1,-2\rangle, \langle 0,0\rangle\rangle\,,$$

where $x \circ_{\gamma_S} y = \gamma_s(x \cdot y)$ and $0^{\boldsymbol{A}_S} = \langle 1,-2\rangle$. The structure of $\boldsymbol{A}_S$ is depicted in Figure 12. Note that $\boldsymbol{A}_S$ is in fact an $\mathrm{FL_{ec}}$-algebra because $x \leq x^2$ holds for every x as the only strictly negative element is $\bot$.

In order to invoke Lemma 5.1.2, we have to show that $\boldsymbol{A}_S$ is strictly simple with a nearly term definable bottom element and $\boldsymbol{A}_S$ is not isomorphic to $\boldsymbol{A}_R$ for $S \neq R$. We start with a lemma showing properties of $0^{\boldsymbol{A}_S}$.

LEMMA 5.3.7. *Let $S \subseteq \mathsf{N}^+$ such that $1 \in S$. Then $\boldsymbol{A}_S$ satisfies for all $k \in \mathsf{N}^+$ the following:*

1. $(0^{\boldsymbol{A}_S})^k = \begin{cases} \langle k,-2\rangle & \text{if } k \in S, \\ \langle k,-1\rangle & \text{otherwise.} \end{cases}$
2. $0^{\boldsymbol{A}_S} \rightarrow_\sigma (0^{\boldsymbol{A}_S})^{k+1} = \langle k,-1\rangle$.

Proof. We will prove the first claim by induction on k. For $k = 1$ the statement holds since $1 \in S$ and $0^{\boldsymbol{A}_S} = \langle 1,-2\rangle$. Assume the validity for k, i.e., $(0^{\boldsymbol{A}_S})^k = \langle k,b\rangle$ for some $b \in \{-1,-2\}$. We have

$$(0^{\boldsymbol{A}_S})^{k+1} = 0^{\boldsymbol{A}_S} \circ_{\gamma_S} (0^{\boldsymbol{A}_S})^k = \gamma_S(\langle 1,-2\rangle \cdot \langle k,b\rangle) = \gamma_S(k+1,-2+b) = \langle k+1,b'\rangle$$

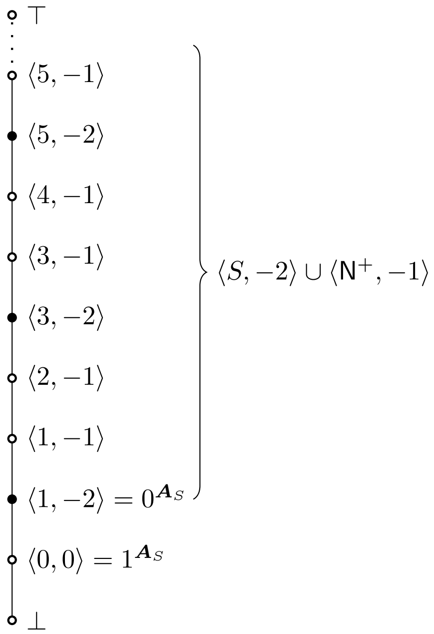

Figure 12. The structure of the algebra $\boldsymbol{A}_S$ for $S = \{1,3,5,\ldots\}$.

for some $b' < 0$. Since $-2+b \leq -2$, we get $b' = -2$ if $k+1 \in S$ and $b' = -1$ otherwise.

Now we prove the second claim. By the first claim we have $(0^{\boldsymbol{A}_S})^{k+1} = \langle k+1, b\rangle$ for some $b \in \{-1,-2\}$. Since $b+2 \geq 0$, we obtain

$$0^{\boldsymbol{A}_S} \to_\sigma (0^{\boldsymbol{A}_S})^{k+1} = \sigma(\langle 1,-2\rangle \to \langle k+1,b\rangle) = \sigma\langle k, b+2\rangle = \langle k,-1\rangle\,.$$

$\square$

LEMMA 5.3.8. *The algebra $\boldsymbol{A}_S$ is strictly simple with a nearly term definable bottom element.*

Proof. The simplicity follows from the fact that each nontrivial filter F in $\boldsymbol{A}_S$ contains $\bot$, i.e., $F = A_S$. Further we claim that $\boldsymbol{A}_S$ is generated by $0^{\boldsymbol{A}_S} = \langle 1,-2\rangle$. Let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}_S$ generated by $0^{\boldsymbol{A}_S}$. First,

$$0^{\boldsymbol{A}_S} \to_\sigma 1^{\boldsymbol{A}_S} = \sigma(\langle 1,-2\rangle \to \langle 0,0\rangle) = \sigma(-1,2) = \bot\,.$$

Thus $\bot \in B$. Second, $\bot \to \bot = \top \in B$. Third, $\langle k,-2\rangle \in B$ for $k \in S$ by Lemma 5.3.7. Finally, $\langle k,-1\rangle \in B$ for any $k \in \mathsf{N}^+$ by the same lemma. Thus $B = A_S$, i.e., $\boldsymbol{A}_S$ is generated by $0^{\boldsymbol{A}_S}$. Consequently, $\boldsymbol{A}_S$ is strictly simple. The bottom element $\bot$ is nearly term definable by the term $x \wedge (x \to 1)$ because $\langle a,b\rangle \to_\sigma \langle 0,0\rangle = \sigma(-a,-b) = \bot$ for $\langle a,b\rangle > \langle 0,0\rangle$. $\square$

It follows from Lemma 5.1.2 that each $\boldsymbol{A}_S$ generates an atom in $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{ec}})$. Thus it suffices to prove that $\boldsymbol{A}_R$ and $\boldsymbol{A}_S$ are not isomorphic for $R \neq S$.

LEMMA 5.3.9. *Let $R, S \subseteq \mathsf{N}^{+}$ such that $R \neq S$ and $1 \in R \cap S$. Then $\boldsymbol{A}_R$ and $\boldsymbol{A}_S$ are not isomorphic.*

Proof. Suppose that $f\colon A_R \to A_S$ is an isomorphism. Take the least element k where R and S differ. Without any loss of generality assume $k \in R$ and $k \notin S$. Then $(0^{\boldsymbol{A}_R})^k = \langle k, -2\rangle$, $(0^{\boldsymbol{A}_S})^k = \langle k, -1\rangle$. Moreover, it follows from Lemma 5.3.7 that

$$\begin{aligned} 0^{\boldsymbol{A}_R} \to_\sigma (0^{\boldsymbol{A}_R})^{k+1} &= \langle k, -1\rangle\,, \\ 0^{\boldsymbol{A}_S} \to_\sigma (0^{\boldsymbol{A}_S})^{k+1} &= \langle k, -1\rangle\,. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & f(k,-1) = f(0^{\boldsymbol{A}_R} \to_\sigma (0^{\boldsymbol{A}_R})^{k+1}) = \\ & \qquad 0^{\boldsymbol{A}_S} \to_\sigma (0^{\boldsymbol{A}_S})^{k+1} = (0^{\boldsymbol{A}_S})^k = f((0^{\boldsymbol{A}_R})^k) = f(k,-2)\,. \end{aligned}$$

Thus f is not one-to-one, so cannot be an isomorphism. □

THEOREM 5.3.10. *There are $2^{\aleph_0}$ atoms in $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{ec}})$.*

To complete Figures 9 and 10, we have to discuss the cardinality of atoms in the subvariety lattices for $\mathrm{RL}^{\ell}_{\mathrm{c}}$-algebras and $\mathrm{FL}^{\ell}_{\mathrm{co}}$-algebras. Continuum many atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{c}})$ were constructed in [16]. In fact, [16] constructs continuum many idempotent RL_{c}-chains (i.e., RL_{c}-chains satisfying $x = x^2$) generating atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{c}})$. As shown in [18] the construction can be further modified in order to prove that also $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{co}})$ contains continuum many atoms. The proofs of all these results are rather involved, so we will state here the next theorem without a proof.

THEOREM 5.3.11. *There are $2^{\aleph_0}$ atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{c}})$ and $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{co}})$.*

6 Historical remarks and further reading

Fuzzy logics as formal logical systems started to be investigated in 1990's. The main contribution in this direction is Hájek's monograph [23]. This book introduces syntactical calculi for fuzzy logics as well as their corresponding algebraic semantics. Later it turned out that fuzzy logics can be viewed as a part of much broader class of so-called substructural logics, i.e., logics lacking some of the structural rules of contraction, exchange, left and right weakening. Hence we present fuzzy logics (i.e., in our sense semilinear substructural logics) inside the hierarchy of substructural logics. For details on substructural logics see the recent book [18] or [34, 35].

Our presentation of FL-algebras which form an equivalent algebraic semantics for the base substructural logic FL, mainly follows [18]. It differs slightly in the section on the Dedekind-MacNeille completion of an FL-algebra. Namely, in the construction of the completion of an FL-algebra $\boldsymbol{A}$ we use a downset monoid $\mathcal{D}(\boldsymbol{A})$ (i.e., an FL-algebra on the set of all downsets) whereas [18] replaces the downset monoid with a powerset monoid $\mathcal{P}(\boldsymbol{A})$ (i.e., an FL-algebra on the set of all subsets). We choose this approach because $\mathcal{P}(\boldsymbol{A})$ is unnecessarily big in comparison with $\mathcal{D}(\boldsymbol{A})$. Consequently, the map $x \mapsto \downarrow x$ is an embedding of the $\vee$-free reduct of $\boldsymbol{A}$ into $\mathcal{D}(\boldsymbol{A})$ (see Proposition 3.8.3) which even preserves $\vee$ if $\boldsymbol{A}$ is linearly ordered. On the other hand, an analogous map from $\boldsymbol{A}$ to $\mathcal{P}(\boldsymbol{A})$ given by $x \mapsto \{x\}$ preserves only the monoid structure of $\boldsymbol{A}$. This modification also allows us to obtain the structural characterization for FL-chains (see Theorem 3.8.11). Concerning the construction of the downset monoid $\mathcal{D}(\boldsymbol{A})$, it is good to mention that it can be further generalized as was known in category theory already in 1970's. Namely, the requirement that $\boldsymbol{A}$ has to be a (pointed) pomonoid can be weakened in order to obtain an FL-algebra by the above construction. It suffices that $\boldsymbol{A}$ is a poset endowed with a *promonoidal* structure (for details see [12]). Further, it is interesting to note that the construction of the Dedekind MacNeille completion can be put into a more general framework of residuated frames (see [17]). Using this framework it was described which identities are preserved by the Dedekind-MacNeille completion (see [7, 8, 39]).

The section on completeness properties with respect to distinguished semantics contains results scattered in various papers. In the literature the completeness property with respect to the class of respective chains on $[0,1]$ is often called *standard completeness*. The proofs of standard completeness presented here are completely new providing hopefully a better insight into matters. The original proofs are usually based on a method invented in [29] proving the strong completeness property of $\mathbb{FL}^{\ell}_{\mathrm{ew}}$ with respect to the class of respective chains on $[0,1]$. This method was further modified by other authors. In particular, [15] proves the standard completeness for $\mathbb{InFL}^{\ell}_{\mathrm{ew}}$ and [30] for $\mathbb{FL}^{\ell}_{\mathrm{w}}$. An analogous result for $\mathbb{FL}^{\ell}_{\mathrm{cw}}$ goes back to 1950's (see [13]). On the other hand, the fact that $\mathbb{FL}^{\ell}$ does not enjoy the standard completeness was proved in [41]. Here we provide a shorter proof together with new results that the same holds also for $\mathbb{FL}^{\ell}_{\mathrm{S}}$ if $\mathrm{S} \subseteq \{\mathrm{c}, \mathrm{o}\}$ and for $\mathbb{InFL}^{\ell}_{\mathrm{S}}$, $\mathbb{CyInFL}^{\ell}_{\mathrm{S}}$ if $\mathrm{S} \subseteq \{\mathrm{c}\}$. Finally, as we mentioned already inside this chapter, the standard completeness for $\mathbb{FL}^{\ell}_{\mathrm{e}}$ was proved in [32] but the used method is proof-theoretical. As far as we know there is no algebraic proof of this fact. The finite strong completeness property of a variety with respect to the class of all finite algebras is known in the literature under the name *finite embeddability property* (FEP). Note that for semilinear varieties the FEP is equivalent to the finite strong completeness property with respect to the class of all finite chains, i.e., $\mathrm{FS}\mathbb{FC}$. The proofs of the FEP are usually based on a method invented in [2], where it is proved that $\mathbb{FL}_{\mathrm{ew}}$ has the FEP. A semilinear modification for $\mathbb{FL}^{\ell}_{\mathrm{ew}}$ and $\mathbb{InFL}^{\ell}_{\mathrm{ew}}$ was given in [9]. Further modifications can be found in [3, 40, 37, 38, 39, 18]. In this chapter we provide a different presentation based on conuclei and nuclei which works for semilinear varieties. For other varieties a related approach using residuated frames can be used (see [17, 20]). The quasi-identity (21) used in the proof of Theorem 4.2.7, showing that several non-integral semilinear varieties do not possess the FEP, was introduced in [2]. An overview of many known results on completeness properties of fuzzy logics can be found in [10].

We should point out that there are other results in the literature on (semilinear) varieties of FL-algebras not mentioned in this chapter. Let us recall a few of them. Likely the most studied variety of (semilinear) FL-algebras not covered in this chapter is the variety of n-potent FL-algebras, i.e., FL-algebras satisfying $x^{n+1} = x^{n}$. In particular, for $n = 2$ we obtain the class of idempotent FL-algebras. Note that idempotent $\mathrm{FL_i}$-algebras coincide with $\mathrm{FL_{ci}}$-algebras. The constructions for proving various completeness properties usually preserve the n-potency identity. Thus one can obtain analogous completeness properties we presented here also for n-potent FL-algebras. An overview for integral semilinear varieties can be found in [10]. The standard completeness for the variety of idempotent $\mathrm{FL}^{\ell}_{\mathrm{e}}$-algebras was proved in [32]. Other well-known identities studied in the realm of FL-algebras are the identities $x\backslash xy = y = yx/x$ expressing the fact the monoid reduct is cancellative. These identities are not as easily preserved as the n-potency. Hence one has to use a different method to prove a completeness property. The finite strong completeness property for the variety of cancellative $\mathrm{RL}^{\ell}_{\mathrm{ei}}$-algebras with respect to the class of chains on $(0, 1]$ was proved in [26]. In fact, [26] contains a slightly different result from which the above-mentioned result easily follows. Namely, [26] proves the finite strong completeness property with respect to the class of chains on $[0, 1]$ for a subvariety of $\mathbb{FL}^{\ell}_{\mathrm{ew}}$ axiomatized by $\neg x \vee ((x \to xy) \to y) = 1$. This identity expresses in an $\mathrm{FL_{ew}}$-chain $\boldsymbol{A}$ that $A \setminus \{0\}$ forms a cancellative monoid. Nontrivial cancellative FL-algebras are necessarily infinite so the varieties of cancellative FL-algebras cannot posses the FEP. Hence it is not clear whether their (quasi)-equational theories are decidable or not. In [27] it is proved that the variety of cancellative $\mathrm{RL}^{\ell}_{\mathrm{ei}}$-algebras enjoys the finite strong completeness property with respect to a countable class of chains. Consequently, it is shown that the quasi-equational theory of this variety is decidable. Last but not least are the identities $x[x\backslash(x \wedge y)] = x \wedge y = [(x \wedge y)/x]x$. They are known under the name *divisibility* expressing that $x \le y$ implies $x = yz$ and $x = z'y$ for some z, z'. The divisible RL-algebras are known as GBL-algebras (see [22]). Moreover, the divisible $\mathrm{FL}^{\ell}_{\mathrm{ew}}$-algebras are known as BL-algebras. The variety of BL-algebras enjoys the finite strong completeness property with respect to the class of chains on $[0, 1]$. All details on BL-algebras can be found in Chapter VII.

Finally, we will comment on the section about the subvariety lattice of semilinear FL-algebras. A good overview on the structure of subvariety lattice of FL-algebras can be found in [18]. The useful criterion presented in Lemma 5.1.2 comes from [16]. The easy results on the subvariety lattices with finitely many atoms were known (see [16, 18]). The result showing that there are only two atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{ei}})$ comes from [25]. This paper also shows how to construct continuum many atoms in $\mathbf{\Lambda}(\mathbb{RL}^{\ell}_{\mathrm{e}})$, $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{ei}})$ and $\mathbf{\Lambda}(\mathbb{FL}^{\ell}_{\mathrm{eo}})$. The algebras generating atoms in the subvariety lattice of $\mathrm{FL}^{\ell}_{\mathrm{ec}}$-algebras were presented in [18]. Here we present how these atoms can be constructed from an ℓ-group by means of a conucleus and a nucleus. Continuum many atoms were also constructed for the subvariety lattice of $\mathrm{RL}^{\ell}_{\mathrm{c}}$-algebras in [16]. A modification for $\mathrm{FL}^{\ell}_{\mathrm{co}}$-algebras is presented in [18]. Concerning other results not mentioned in this chapter, it is clear that more is known about subvariety lattices for smaller semilinear varieties of FL-algebras. For instance it is known that the subvariety lattice of idempotent $\mathrm{RL}^{\ell}_{\mathrm{e}}$-algebras is countable (see [33]). A lot is known about the subvariety lattice of BL-algebras. These results are mentioned in Chapter V on MV-algebras and Chapter VII on BL-algebras. There are also results on almost minimal (semilinear) varieties of FL-algebras, i.e., varieties which cover atoms in the subvariety lattice. For details see [28, 31, 18].

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INDEX