First-order definability for distributive modal substructural logics: an indirect method

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Abstract

We investigate the relationship between Kripke semantics and bi-approximation semantics over distributive lattice-based logics such as distributive modal substructural logics. In particular, based on the prime skeletons introduced in [21], we provide a back-and-forth construction between Kripke semantics and bi-approximation semantics. Throughout the procedure, we transfer the Sahlqvist first-order definability shown in [26] to a first-order definability result on Kripke semantics. Finally, we will give a positive answer for an open question raised in [8] as an application of our result.

1 Introduction

Kripke semantics [14] has given a great impact to apply modal logic to other areas of science, for example, computer science. Unlike what happens in the algebraic semantics, the completeness results on Kripke semantics cannot be universally discussed for all axiomatic extensions of normal modal logic $\mathbf{K}$. Even more interestingly, it is also known the existence of incomplete modal logics today. As a natural consequence, we have studied how to account for Kripke complete modal logics, namely modal logics which are complete with respect to Kripke semantics. The so-called canonical model construction has been introduced as a general gateway to explicate Kripke completeness in modal logic. On one hand, the canonical model construction does not give a complete characterisation of Kripke complete modal logics: that is, there are also some Kripke complete modal logics which cannot be explained by the canonical model construction. On the other hand, it has yielded a schematic account of Kripke complete modal logics as follows: with help of Fine’s theorem “every elementary logic is canonical [3],” Sahlqvist has shown an algorithmic method to obtain the first-order correspondents of a certain type of formulae, called Sahlqvist formulae [18], see also e.g. [10]. Today the Sahlqvist-type arguments are not just reformulated, for example in [19, 2], but also applied to other non-classical logics, for example in
[7, 20]. We also note that the algebraic discussion of the canonicity has already been in [13].

In substructural logic, the Sahlqvist-type canonicity results have been obtained for intuitionistic modal logic in [8] as a natural generalisation of the original Sahlqvist theorem in modal logic, and recently it has also been generalised to lattice-based logics in [23] and to poset-based logics in [24] as well. These results claim that every sequent (formula) which has consistent variable occurrence is canonical. However, unlike what happens in the setting of modal logic, the first-order definability on Kripke semantics for intuitionistic modal logic, or more generally Kripke semantics for distributive modal substructural logics, is not thoroughly discussed yet as is mentioned in [8].

In [22], the author has introduced a relational-type semantics, named bi-approximation semantics, for substructural logic as a polarity-based relational semantics for non-distributive, i.e. not necessarily distributive, lattice-based logics. We also mention that, in the literature, other discussions of a relational semantics for non-distributive lattice-based logics can be found, for example, in [9, 11, 12, 17, 6, 4]. As distinct from Kripke semantics, [26] has shown that the first-order definability of all sequents which have consistent variable occurrence, holds on bi-approximation semantics for substructural logic. Therefore it can be a completion of the Sahlqvist theorem for substructural logic. On the other hand, as is mentioned in [8], the first-order definability on Kripke semantics is still unknown: even for intuitionistic modal logic, it is not clear yet whether all sequents with consistent variable occurrence have first-order correspondents on Kripke semantics. Hence, in the current paper, we shall discuss the first-order definability for distributive modal substructural logics on Kripke semantics.

In the recent study of bi-approximation semantics [25, 21], we have studied how polarity frames, which are the basements of bi-approximation semantics, accept the distributive law $\phi \land (\psi \lor \chi) \Rightarrow (\phi \land \psi) \lor (\phi \land \chi)$, and compared them with Kripke semantics. In this paper, we extend the results to additional relational structures on bi-approximation semantics and show a back-and-forth construction between Kripke semantics and bi-approximation semantics over distributive modal substructural logics. As a result, we notice that the first-order definability on bi-approximation semantics [26] is transferable to the one on Kripke semantics. Therefore we shall return a positive answer for the open question in [8].

We outline the structure of the paper as follows: in Section 2, we introduce the distributive modal substructural logic DFLM with the sequent system, which is set up as a natural generalisation of intuitionistic modal logic in [8] to apply our results in Section 6, and recall the basic results of substructural logic. Next, in Section 3, Kripke-type semantics of distributive modal substructural logics is spelled out. In the paper, we formulate the semantics to compare directly with bi-approximation semantics of distributive modal substructural logics. Section 4 shortly recalls bi-approximation semantics of distributive modal substructural logics and the fundamental properties. Additionally, we also discuss how essentially splitters work on bi-approximation semantics of distributive modal substructural logics. Then we investigate invariance of va-
lidity of sequents between Kripke semantics and bi-approximation semantics by means of the results in [25, 21] in Section 5. Based on the invariance of validity of sequents and a back-and-forth construction between Kripke semantics and bi-approximation semantics, we prove the first-order definability on Kripke semantics. In Section 6, we apply our results to intuitionistic modal logic and show the first-order definability on Kripke semantics, which gives rise to a positive answer for the open question in [8]. In the end, we summarise our results in Section 7.

2 Distributive modal substructural logics

In the literature, there are still a lot of interesting discussions of substructural logics with modalities, for example see [17]. However, in order to discuss a Sahlqvist-type first-order correspondence theory and apply it for the intuitionistic modal logic in [8], we introduce distributive modal substructural logics in the following manner.

The language of distributive modal substructural logics consists of propositional variables \( p, q, \ldots \), two constants \( t \) and \( f \), and six logical connectives disjunction \( \lor \), conjunction \( \land \), fusion \( \circ \), implications \( \rightarrow \) and \( \leftarrow \), diamond \( \Diamond \) and box \( \Box \). We denote by \( \Phi \) the set of all propositional variables. In this language, a modal substructural formula \( \phi \), or a formula for short, is defined by the following Backus-Naur form:

\[
\phi ::= p \mid t \mid f \mid \phi \lor \phi \mid \phi \land \phi \mid \phi \circ \phi \mid \phi \rightarrow \phi \mid \phi \leftarrow \phi \mid \Diamond \phi \mid \Box \phi ,
\]

where \( p \in \Phi \). We denote by \( \Lambda \) the set of all formulae. For a finite list of formulae \( \phi_1, \ldots, \phi_m \) and at most one formula \( \phi \), the pair is denoted by \( \phi_1, \ldots, \phi_m \implies \phi \) and it is called a sequent.

The distributive modal substructural logic DFLM. Let \( \phi, \psi, \chi \) be arbitrary formulae, \( \Gamma, \Delta, \Sigma, \Pi \) arbitrary (possibly empty) finite lists of formulae, \( \varphi \) a list of at most one formula. We define the distributive full Lambek calculus with modalities DFLM as in Figure 1. In the sequent system DFLM, a formula \( \phi \) is **provable in DFLM** if the sequent \( \Gamma \implies \phi \) is derivable in DFLM. The distributive modal substructural logic DFLM is the set of all provable formulae in DFLM. We can obtain the following results for substructural logics: for example, see [16].

**Proposition 2.1.** For all formulae \( \phi, \phi_1, \ldots, \phi_n, \varphi \) and \( \psi \), we have

1. \( \phi \) is provable if and only if \( t \implies \phi \) is derivable,

2. \( \phi \implies \psi \) is derivable if and only if \( \phi \implies f \) is derivable,

3. \( \phi \implies \psi \) is derivable if and only if \( \phi \rightarrow \psi \) is provable in DFLM if and only if \( \psi \leftarrow \phi \) is provable in DFLM,
Initial sequents.

\[ \phi \vdash \phi \quad \Rightarrow t \quad \Gamma \vdash (\phi \land (\psi \lor \chi)) \vdash (\phi \land (\psi \lor (\phi \land \chi))) \]

Cut rule.

\[
\frac{\Gamma \vdash \phi \quad \Sigma, \phi, \Pi \vdash \varphi}{\Sigma, \Gamma, \Pi \vdash \varphi} \quad \text{(cut)}
\]

Rules for logical constants.

\[
\frac{\Gamma, \Delta \vdash \varphi}{\Gamma, t, \Delta \vdash \varphi} \quad \text{(tw)} \quad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi} \quad \text{(fW)}
\]

Rules for logical connectives.

\[
\frac{\Gamma, \phi, \Delta \vdash \varphi \quad \Gamma, \psi, \Delta \vdash \varphi}{\Gamma, \phi \lor \psi, \Delta \vdash \varphi} \quad \text{(\lor \vdash)}
\]

\[
\frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \lor \psi} \quad \text{($\lor_1$)} \quad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \lor \psi} \quad \text{($\lor_2$)}
\]

\[
\frac{\Gamma, \phi, \Delta \vdash \varphi}{\Gamma, \phi \land \psi, \Delta \vdash \varphi} \quad \text{($\land_1$)} \quad \frac{\Gamma, \psi, \Delta \vdash \varphi}{\Gamma, \phi \land \psi, \Delta \vdash \varphi} \quad \text{($\land_2$)}
\]

\[
\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \land \psi} \quad \text{($\land$)}
\]

\[
\frac{\Gamma, \phi, \psi, \Delta \vdash \varphi}{\Gamma, \phi \circ \psi, \Delta \vdash \varphi} \quad \text{($\circ$)} \quad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \circ \psi} \quad \text{($\circ$)}
\]

\[
\frac{\Gamma \vdash \phi}{\Sigma, \phi \rightarrow \psi, \Pi \vdash \varphi} \quad \text{($\rightarrow \vdash$)} \quad \frac{\phi, \Gamma \vdash \psi}{\Gamma \vdash \phi \vdash \psi} \quad \text{($\rightarrow$)}
\]

\[
\frac{\Gamma \vdash \phi}{\Sigma, \psi \leftarrow \phi, \Gamma, \Pi \vdash \varphi} \quad \text{($\leftarrow \vdash$)} \quad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \psi \leftarrow \phi} \quad \text{($\leftarrow$)}
\]

\[
\frac{\Diamond \phi \vdash \psi}{\phi \vdash \Box \psi} \quad \text{($\vdash \Box$)} \quad \frac{\phi \vdash \Box \psi}{\Diamond \phi \vdash \psi} \quad \text{($\Box$)}
\]

Figure 1: The distributive full Lambek calculus DFLM.
$\phi_1, \ldots, \phi_n \Rightarrow \varphi$ is derivable in DFLM if and only if
$\phi_1 \circ \cdots \circ \phi_n \Rightarrow \varphi$ is derivable in DFLM.

Proposition 2.1, allows us to think about every sequent equivalently of a pair of two formulae $\phi \Rightarrow \psi$. Due to the fact, hereinafter, we may call a pair of two formulae a sequent. Furthermore, we sometimes state that DFLM is the set of all sequents derivable in DFLM.

**Proposition 2.2.** The following two rules are derivable in DFLM.

$\phi \Rightarrow \psi$

$\square \phi \Rightarrow \square \psi$ (□)

$\diamond \phi \Rightarrow \diamond \psi$ (◊)

Furthermore, the following four sequents are derivable in DFLM.

1. $\square (\phi \land \psi) \Rightarrow \square \phi \land \square \psi$,
2. $\square \phi \land \square \psi \Rightarrow \square (\phi \land \psi)$,
3. $\diamond \phi \lor \diamond \psi \Rightarrow \diamond (\phi \lor \psi)$,
4. $\diamond (\phi \lor \psi) \Rightarrow \diamond \phi \lor \diamond \psi$.

**Proof.** Here we show only the proofs of item 2 and item 3 in Figure 2 and Figure 3, respectively. But item 1 and item 4 are also analogously provable.

Notice that the rule (◊) is actually in the second proof, and the rule (□) is

$\square (\phi \land \psi) \Rightarrow \square \phi \land \square \psi$ (□)

$\phi \Rightarrow \psi$

$\diamond (\square \phi \land \square \psi) \Rightarrow \diamond (\phi \land \psi)$ (⇒)

$\diamond \phi \land \diamond \psi \Rightarrow \diamond (\phi \land \psi)$ (⇒)

$\square (\phi \land \psi) \Rightarrow \square \phi \land \square \psi$ (⇒)

$\phi \Rightarrow \psi$

$\diamond (\phi \land \psi) \Rightarrow \diamond (\phi \land \psi)$ (⇒)

$\square \phi \land \square \psi \Rightarrow \square (\phi \land \psi)$ (⇒)

$\diamond \phi \land \square \psi \Rightarrow \square (\phi \land \psi)$ (⇒)

Figure 2: A proof for item 2

We call an axiomatic extension of DFLM (as a collection of sequents or formulae) a distributive modal substructural logic. The algebraic counterparts of substructural logic FL are known as FL-algebras: see [5]. As an expansion, we define the algebraic counterparts of the distributive modal substructural logic DFLM as follows.

**Definition 2.3** (Distributive full Lambek modal algebra). A decuple $\mathbb{A} = (A, \lor, \land, *, \backslash, /, \diamond, \square, 1, 0)$ is a distributive full Lambek modal algebra, a DFLM-algebra for short, if $(A, \lor, \land, *, \backslash, /, 1, 0)$ is an FL-algebra satisfying the distributive law, and $\diamond$ and $\square$ are unary monotone functions on $A$ which satisfy the following adjointness condition: for all $a, b \in A$,

$\diamond a \leq b \iff a \leq \square b$. 

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Remark 2.4. The class of DFLM-algebras is equationally definable. More precisely, it is defined by equations for FL-algebras with the following conditions: for all $a, b, c \in A$,

1. $a \land (b \lor c) = (a \land b) \lor (a \land c)$ or equivalently $(a \lor b) \land (a \lor c) = a \lor (b \land c)$,
2. $\Diamond (a \lor b) = \Diamond a \lor \Diamond b$,
3. $\Box (a \land b) = \Box a \land \Box b$,
4. $a \lor \Box \Diamond a = a$ or equivalently $a \lor \Box \Diamond a = \Box \Diamond a$,
5. $a \lor \Diamond \Box a = a$ or equivalently $a \lor \Diamond \Box a = \Diamond \Box a$.

On DFLM-algebras, each formula $\phi$ is interpreted to the corresponding term $t$ as usual, that is, $0, ←, →, \top$ and $f$ are interpreted to $*, \land, \lor, 1$ and $0$.

Theorem 2.5. The distributive modal substructural logic DFLM is sound and complete with respect to the class of DFLM-algebras.

3 Kripke-type semantics of distributive modal substructural logics

As a relational semantics of distributive modal substructural logics, or a Kripke-type semantics of distributive modal substructural logics, there are several types of relational frameworks spelled out via the Stone representation for bounded distributive lattices. But, in order to have a straight comparison to bi-approximation semantics, we introduce a relational semantics of distributive modal substructural logics in the following form.
Definition 3.1. A sextuple \( \mathbb{K} = (W, \preceq, R_o, S_o, O, N) \) is a Kripke frame of distributive modal substructural logics, a Kripke frame for short, if \( W \) is a non-empty set, \( \preceq \) is a preorder on \( W \), \( R_o \) is a ternary relation on \( W \), \( S_o \) is a binary relation on \( W \), \( O \) is a non-empty downward closed subset of \( W \), \( N \) is a downward closed subset of \( W \), and \( \mathbb{K} \) satisfies

- **\( R_o \)-order**: \( w_i \preceq w_j \) if and only if \( \exists o \in O. [R_o(w_j, o, w_i) \text{ or } R_o(o, w_j, w_i)] \),
- **\( R_o \)-identity**: \( \exists o_1 \in O. [R_o(o_1, w_i, w_i)] \) and \( \exists o_2 \in O. [R_o(w_i, o_2, w_i)] \),
- **\( R_o \)-transitivity**:
  \[ w_i \preceq w'_i, w_j \preceq w'_j, w'_k \preceq w_k \text{ and } R_o(w_i, w_j, w_k) \text{ imply } R_o(w'_i, w'_j, w'_k), \]
- **\( R_o \)-associativity**: \( \exists v \in W. [R_o(w_i, v, w)] \) and \( R_o(w_j, w_k, v) \) if and only if \( \exists u \in W. [R_o(u, w_k, v) \text{ and } R_o(w_i, w_j, u)] \),
- **\( S_o \)-transitivity**: \( w_i \preceq w'_i, w'_j \preceq w_j \text{ and } S_o(w_i, w_j) \text{ imply } S_o(w'_i, w'_j) \),

where all (decorated) \( w \) are universally quantified over the whole sentence in each case and range over \( W \).

Note that we take the order dual definition of normal Kripke frames of distributive modal substructural logics. Usually, we take upward closed subsets of \( W \) as \( O \) and \( N \), but here we take downward closed subsets. Plus, the \( R_o \) and \( S_o \) conditions are also dually-ordered.

Given a Kripke frame \( \mathbb{K} = (W, \preceq, R_o, S_o, O, N) \), a *valuation* \( V_K \) on \( \mathbb{K} \) is a function from \( \Phi \) to the set \( D_K \) of all downsets of \( (W, \preceq) \). We call a pair of a Kripke frame \( \mathbb{K} \) and a valuation \( V_K \) on \( \mathbb{K} \) a *Kripke model*. On a Kripke model \( \mathcal{M}_K = (\mathbb{K}, V_K) \), we evaluate formulae as follows: for each \( w \in W \), we let

1. \( \mathcal{M}_K, w \models p \iff w \in V_K(p) \) for each propositional variable \( p \in \Phi \),
2. \( \mathcal{M}_K, w \models \top \iff w \in O \),
3. \( \mathcal{M}_K, w \models \bot \iff w \in N \),
4. \( \mathcal{M}_K, w \models \phi \lor \psi \iff \mathcal{M}_K, w \models \phi \text{ or } \mathcal{M}_K, w \models \psi \),
5. \( \mathcal{M}_K, w \models \phi \land \psi \iff \mathcal{M}_K, w \models \phi \text{ and } \mathcal{M}_K, w \models \psi \),
6. \( \mathcal{M}_K, w \models \phi \circ \psi \iff \exists w_1, w_2 \in W. [\mathcal{M}_K, w_1 \models \phi \land \mathcal{M}_K, w_2 \models \psi \text{ and } R_o(w_1, w_2, w)] \),
7. \( \mathcal{M}_K, w \models \phi \rightarrow \psi \iff \forall w_1, w_2 \in W. [\mathcal{M}_K, w_1 \models \phi \land R_o(w_1, w, w_2) \text{ imply } \mathcal{M}_K, w_2 \models \psi] \),
8. \( \mathcal{M}_K, w \models \psi \leftarrow \phi \iff \forall w_1, w_2 \in W. [\mathcal{M}_K, w_1 \models \phi \land R_o(w, w_1, w_2) \text{ imply } \mathcal{M}_K, w_2 \models \psi] \),
9. \( \mathcal{M}_K, w \models \lozenge \phi \iff \exists w' \in W. [S_o(w', w) \text{ and } \mathcal{M}_K, w' \models \phi] \).
10. \( M_K, w \vDash \Box \phi \iff \forall w' \in W. \left[ S_\phi(w, w') \Rightarrow M_K, w' \vDash \phi \right] \).

**Remark 3.2.** One may feel that the satisfaction relation for \( \Diamond \) is not natural. But, this is just because our modalities \( \Diamond \) and \( \Box \) do not have the duality in the normal sense of modal logic. That is, in modal logic, the modalities have the interdefinability by means of the classical negation, i.e. \( \Diamond \) is the same as \( \neg \Box \neg \), or \( \Box \) is the same as \( \neg \Diamond \neg \). However, in our setting (and in [8]), unlike what happens in the setting of modal logic, the modalities have the adjointness only. Therefore we may think that \( \Box \) means the so-called “future necessity” and \( \Diamond \) does the so-called “past possibility.”

Due to the generality of our setting, the truth condition is also reformulated as follows: when \( w \in O \), we interpret \( M_K, w \vDash \phi \) as “a formula \( \phi \) is true at \( w \),” and denote by \( M_K, w \vDash_1 \phi \) for all \( w \in O \), we say that the formula \( \phi \) is *universally true on* \( M_K \), denoted by \( M_K \vDash_1 \phi \). Furthermore, if a formula \( \phi \) is universally true on \( \langle K, V_K \rangle \) for every valuation \( V_K \in D_K \), we say that the formula \( \phi \) is *valid on* \( K \), denoted by \( K \vDash_1 \phi \). Based on this evaluation, we can prove directly the soundness and the completeness results of DFLM for Kripke frames. However, in this paper, we shall induce not only the first-order definability but all the necessary results through a back-and-forth construction between Kripke semantics and bi-approximation semantics in the subsequence sections.

## 4 Distributive polarity frames for bi-approximation semantics

In this section, we shortly recall the discussion of the distributivity on polarity frames in [21], and extend the results to the bi-approximation semantics for distributive modal substructural logics.

A *polarity frame* is a triple \( \langle X, Y, B \rangle \) of two non-empty sets \( X \) and \( Y \), and a binary relation \( B \) on \( X \times Y \), that is, \( B \subseteq X \times Y \). If a polarity frame satisfies \( B = X \times Y \), it is called *trivial*.

Given a polarity frame \( \langle X, Y, B \rangle \), the binary relation \( B \) is naturally extended to a preorder \( \leq_B \) on \( X \cup Y \) as follows: for all \( x, x_1, x_2 \in X \) and \( y, y_1, y_2 \in Y \),

1. \( x_1 \leq_B x_2 \iff \forall y' \in Y. \left[ x_2 B y' \Rightarrow x_1 B y' \right], \)
2. \( y_1 \leq_B y_2 \iff \forall x' \in X. \left[ x' B y_1 \Rightarrow x' B y_2 \right], \)
3. \( x \leq_B y \iff x B y, \)
4. \( y \leq_B x \iff \forall x' \in X, y' \in Y. \left[ x B y' \text{ and } x' B y \Rightarrow x' B y' \right]. \)

We may omit the subscript \( \_B \) when is clear from the context, and we may sometimes call a triple \( \langle X, Y, \leq \rangle \) a polarity frame, instead of \( \langle X, Y, B \rangle \). An element \( x \in X \) is a *splitter*, if there exists a splitting counterpart \( y_s \in Y \) of \( x \) such that...
1. \( x \not\leq y \), and
2. \( \forall y \in Y. [x \not\leq y \implies y \leq y'] \).

Analogously, an element \( y \in Y \) is a splitter, if there exists a splitting counterpart \( x_y \in X \) of \( y \) such that
1. \( x_y \not\leq y \) and
2. \( \forall x \in X. [x \not\leq y \implies x_y \leq x] \).

Note that every splitting counterpart is also a splitter as well. Moreover, splitting counterparts of a splitter are unique up to \( \leq \)-equivalence, namely if \( y_x \) and \( y'_x \) are splitting counterparts of a splitter \( x \) then we have \( y_x \leq y'_x \) and \( y'_x \leq y_x \).

By expanding distributive polarity frames, we obtain polarity frames for distributive modal substructural logics as follows.

**Definition 4.1** (Distributive polarity frame). A polarity frame \( (X, Y, B) \) is distributive, for all \( x \in X \) and \( y \in Y \), if \( xBy \) does not hold, there exists a splitting pair \((x, y_x)\) such that \( x_x \leq_B x \) and \( y \leq_B y_x \). Furthermore, a polarity frame is prime, if all elements in \( X \) and in \( Y \) are splitters.

Although trivial polarity frames are also distributive by definition, there is no splitting pair. It causes a technical problem later. Therefore, henceforth we restrict our attention only on non-trivial distributive polarity frames. Also, notice that every prime polarity frame is distributive by definition but the converse does not hold in general.

By expanding distributive polarity frames, we obtain polarity frames for distributive modal substructural logics as follows.

**Definition 4.2** (Polarity frame of distributive modal substructural logics). A polarity frame of distributive modal substructural logics, a p-frame for short, is a nonuple \( F = (X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y) \), if \( (X, Y, \leq) \) is a non-trivial distributive polarity frame, \( R \) is a ternary relation on \( X \times X \times Y \), \( S \) is a binary relation on \( X \times Y \), \( O_X \) is a non-empty subset of \( X \), \( N_X \) is a subset of \( X \), \( O_Y \) is a proper subset of \( Y \), \( N_Y \) is a subset of \( Y \), and \( F \) satisfies

- **R-order**: \( x' \leq x \) if and only if \( \exists o \in O_X. [R^o(x, o, x') \text{ or } R^o(o, x, x')] \),

- **R-identity**: \( \exists o_1 \in O_X. [R^o(o_1, x, x)] \) and \( \exists o_2 \in O_X. [R^o(x, o_2, x)] \),

- **R-transitivity**: \( x'_1 \leq x_1, x'_2 \leq x_2, y \leq y' \) and \( R(x_1, x_2, y) \) imply \( R(x'_1, x'_2, y') \),

- **R-associativity**: \( \exists z \in X. [R^o(x_1, z, x) \text{ and } R^o(x_2, x_3, z)] \) if and only if \( \exists z' \in X. [R^o(z', x_3, x) \text{ and } R^o(x_1, x_2, z')] \),

- **S-transitivity**: \( x' \leq x, y \leq y' \) and \( S(x, y) \) imply \( S(x', y') \),

- **O-isom**: \( O_X = \{ x \in X \mid \forall y \in O_Y. x \leq y \} \)
  and \( O_Y = \{ y \in Y \mid \forall x \in O_X. x \leq y \} \),

- **N-isom**: \( N_X = \{ x \in X \mid \forall y \in N_Y. x \leq y \} \)
  and \( N_Y = \{ y \in Y \mid \forall x \in N_X. x \leq y \} \),

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\(\Diamond\)-tightness: if \(\forall x' \in X. [R^\circ(x_1, x_2, x') \implies x' \leq y] \) then \(R(x_1, x_2, y)\),

\(\rightarrow\)-tightness: if \(\forall y_2 \in Y. [R^- (x_1, y_2, y) \implies x_2 \leq y_2] \) then \(R(x_1, x_2, y)\),

\(\leftarrow\)-tightness: if \(\forall y'_1 \in Y. [R^- (y'_1, x_2, y) \implies x_1 \leq y'_1] \) then \(R(x_1, x_2, y)\),

\(\bigcirc\)-tightness: if \(\forall x' \in X. [S^\bigcirc(x, x') \implies x' \leq y] \) then \(S(x, y)\),

\(\square\)-tightness: if \(\forall y' \in Y. [S^\square(y', y) \implies x \leq y'] \) then \(S(x, y)\),

where all (decorated) \(x\) and \(y\) range over \(X\) and \(Y\), they are universally quantified over the whole sentence in each case, also the auxiliary relations \(R^\circ(x_1, x_2, x)\), \(R^\rightarrow(x_1, y_2, y)\), \(R^- (y_1, x_2, y)\), \(S^\bigcirc(y', y)\) and \(S^\square(y', y)\) are abbreviations of \(\forall y' \in Y. [R(x_1, x_2, y') \implies x \leq y']\), \(\forall x'_2 \in X. [R(x_1, x_2, y) \implies x'_2 \leq y_2]\), \(\forall x'_1 \in X. [R(x'_1, x_2, y) \implies x'_1 \leq y_1]\), \(\forall y' \in Y. [S(x, y') \implies x' \leq y']\) and \(\forall x' \in X. [S(x', y) \implies x' \leq y']\), respectively.

**Remark 4.3.** Unlike \(p\)-frames of substructural logics in [22, 26], we restrict \(O_Y\) as a proper subset of \(Y\), i.e. \(O_Y \neq Y\), here. This restriction allows us to discuss the prime skeletons introduced in the next section smoothly, see Theorem 5.1. However, the limitation does not affect anything significant, because the \(p\)-frames with \(O_Y = Y\) are trivial polarity frames, which means that those \(p\)-frames validate the sequent \(t \Rightarrow \phi\) for each formula \(\phi \in \Lambda\), hence every sequent is valid by Proposition 2.1, see also Remark 4.10.

**Remark 4.4.** By the definitions of \(R^\circ\), \(R^\rightarrow\), \(R^-\), \(S^\bigcirc\) and \(S^\square\), and the transitivity of \(R\) and \(S\), we also have the following transitivity for \(R^\circ\), \(R^\rightarrow\), \(R^-\), \(S^\bigcirc\) and \(S^\square\): for all \(x_1, x_2, x, x'_1, x'_2, x' \in X\) and \(y_1, y_2, y, y'_1, y'_2, y' \in Y\),

1. \(x_1 \leq x'_1, x_2 \leq x'_2, x' \leq x\) and \(R^\circ(x_1, x_2, x')\) imply \(R^\circ(x'_1, x'_2, x')\),

2. \(x_1 \leq x'_1, y_2 \leq y'_2, y' \leq y\) and \(R^- (x_1, y_2, y)\) imply \(R^- (x'_1, y'_2, y')\),

3. \(y_1 \leq y'_1, x_2 \leq x'_2, y' \leq y\) and \(R^- (y_1, x_2, y)\) imply \(R^- (y'_1, x'_2, y')\),

4. \(x_1 \leq x'_1, x' \leq x\) and \(S^\bigcirc(x_1, x)\) imply \(S^\bigcirc(x'_1, x')\),

5. \(y_1 \leq y'_1, y' \leq y\) and \(S^\square(y_1, y)\) imply \(S^\square(y'_1, y')\).

On a \(p\)-frame \(F\), we introduce a special valuation as follows. A pair \(V = (V^+, V^-)\) of functions \(V^+: \Phi \to \wp(X)\) and \(V^-: \Phi \to \wp(Y)\), where \(\wp(X)\) is the powerset of \(X\) and \(\wp(Y)\) is the powerset of \(Y\), is a doppelgänger valuation on \(F\), if, for each propositional variable \(p \in \Phi\),

1. \(V^+(p) = \{x \in X \mid \forall y \in V^+(p), x \leq y\}\),

2. \(V^-(p) = \{y \in Y \mid \forall x \in V^+(p), x \leq y\}\).

We denote by \(D_V\) the set of all doppelgänger valuation on \(F\). We call a pair of a \(p\)-frame \(F\) and a doppelgänger valuation \(V\) on \(F\), denoted by \(M = (\wp(F), V)\), a bi-approximation model. On a bi-approximation model \(M = (\wp(F), V)\), we introduce a satisfaction relation \(\models\) as follows: for each \(x \in X\) and each \(y \in Y\),

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First-order definability for DFLM (submitted to Annals of Pure and Applied Logic)

X-1. $M \models p \iff x \in \text{V}_1(p)$ for each propositional variable $p \in \Phi$,
X-2. $M \models t \iff x \in \text{O}_X$,
X-3. $M \models f \iff x \in \text{N}_X$,
X-4. $M \models \phi \lor \psi \iff \forall y' \in Y. [M \models \phi \lor \psi \Rightarrow x \leq y']$,
X-5. $M \models \phi \land \psi \iff M \models \phi$ and $M \models \psi$,
X-6. $M \models \phi \circ \psi \iff \forall y' \in Y. [M \models \phi \circ \psi \Rightarrow x \leq y']$,
X-7. $M \models \phi \Rightarrow \psi \iff$
\[ \forall x_1 \in X, y' \in Y. [M \models \phi \Rightarrow \psi \Rightarrow R(x_1, x, y')] \]
X-8. $M \models \psi \leftarrow \phi \iff$
\[ \forall x_2 \in X, y' \in Y. [M \models \phi \leftarrow \psi \Rightarrow R(x, y', x_2, y')] \]
X-9. $M \models \bigcirc \phi \iff \forall y' \in Y. [M \models \bigcirc \phi \Rightarrow \psi \Rightarrow x \leq y']$,
X-10. $M \models \Box \phi \iff \forall y' \in Y. [M \models \Box \phi \Rightarrow S(x, y')]$,
Y-1. $M \models_v p \iff y \in \text{V}_1(p)$ for each propositional variable $p \in \Phi$,
Y-2. $M \models_v t \iff y \in \text{O}_Y$,
Y-3. $M \models_v f \iff y \in \text{N}_Y$,
Y-4. $M \models_v \phi \lor \psi \iff M \models_v \phi$ and $M \models_v \psi$,
Y-5. $M \models_v \phi \land \psi \iff \forall x' \in X. [M \models_v \phi \land \psi \Rightarrow x' \leq y]$,
Y-6. $M \models_v \phi \circ \psi \iff \forall x_1, x_2 \in X. [M \models_v \phi \circ \psi \Rightarrow R(x_1, x_2, y)]$,
Y-7. $M \models_v \phi \Rightarrow \psi \iff \forall x' \in X. [M \models_v \phi \Rightarrow \psi \Rightarrow x' \leq y]$,
Y-8. $M \models_v \psi \leftarrow \phi \iff \forall x' \in X. [M \models_v \psi \leftarrow \phi \Rightarrow x' \leq y]$,
Y-9. $M \models_v \bigcirc \phi \iff \forall x' \in X. [M \models_v \bigcirc \phi \Rightarrow S(x', y)]$,
Y-10. $M \models_v \Box \phi \iff \forall x' \in X. [M \models_v \Box \phi \Rightarrow x' \leq y]$,
S-1. $M \models_v \phi \Rightarrow \psi \iff$ if $M \models_v \phi$ and $M \models_v \psi$ then $x \leq y$,
Theorem 4.8 (Soundness). The distributive modal substructural logic DFLM is sound for p-frames.
**Theorem 4.9** (Completeness and first-order definability). Let $\Omega$ be the set of sequents which have consistent variable occurrence. The distributive modal substructural logic $\text{DFLM} \oplus \Omega$ extended by $\Omega$ is complete with respect to a class of first-order definable polarity frames of distributive modal substructural logics.

**Remark 4.10.** As we noted in Remark 4.3, unlike in the setting of $p$-frames in [22, 26], $O_Y$ is a proper subset of $Y$. On the other hand, the sequent $t \Rightarrow p$ has consistent variable occurrence, hence, by means of [23, 26], we can prove the completeness of $\text{DFLM} \oplus \{t \Rightarrow p\}$ with respect to a class of first-order definable $p$-frames. In fact, the first-order correspondent of $t \Rightarrow p$ is that $\forall y \in Y, [y \in O_y]$, namely “$O_Y$ is proper.” One may feel that this restriction yields a contradiction at first. However, since the logic $\text{DFLM} \oplus \{t \Rightarrow p\}$ is the set of all sequents (all formulae), the logic is complete with respect to the empty class of $p$-frames. Therefore, the restriction does not lead any contradiction and it consistently works.

On distributive polarity frames, splitters play central roles: that is, [21] has shown that all splitters $x_s$ and $y_s$ on any bi-approximation model $M$, satisfy $M \models^\leq \phi \lor \psi \iff M \models^\leq \phi$ or $M \models^\leq \psi$, and $M \models^\leq \phi \land \psi \iff M \models^\leq \phi$ or $M \models^\leq \psi$. Consequently, the distributivity $\phi \land (\psi \lor \chi) \Rightarrow (\phi \land \psi) \lor (\phi \land \chi)$ is valid on any distributive polarity frame. In fact, we can extend the splitters’ properties on $p$-frames of distributive modal substructural logics to other logical connectives as well. That is, we can state special properties for splitters on the additional structures $R$, $S$, $O_X$, $O_Y$, $N_X$ and $N_Y$ including the auxiliary relations $R^\circ$, $R^\rightarrow$, $R^\leftarrow$, $S^0$ and $S^\ominus$. Before discussing the properties, we recall the following property for splitters on distributive polarity frames, first.

Let $\langle X, Y, \leq \rangle$ be a distributive polarity frame. For any splitter $x_s \in X$, the splitting counterparts of $x_s$ are unique up to $\leq$-equivalence. Analogously, for any splitter $y_s \in Y$, the splitting counterparts of $y_s$ are unique up to $\leq$-equivalence. Because the splitting counterparts of $x_s (y_s)$ are also splitters which have $x_s (y_s)$ as a splitting counterpart, splitters in $X$ and splitters in $Y$ are essentially isomorphic, i.e. the uniqueness is up to $\leq$-equivalence. In other words, there is the correspondence between splitters in $X$ and in $Y$ up to $\leq$-equivalence. Hence, hereinafter, we denote splitters forming a splitting pair by adding the same alphabetical (not numerical) indexes as subscripts, e.g. $x_s$ and $y_s$. If a splitter has more than two splitting counterparts, we randomly choose a representative. Now, we show the crucial properties for splitters on the additional structures of distributive polarity frames.

**Proposition 4.11.** Let $F$ be a $p$-frame. For each splitter $x_s \in X_s$ whose splitting counterpart is $y_s \in Y_s$, each splitter $y_t \in Y$ whose splitting counterpart is $x_t \in X_s$, all $x, x_1, x_2 \in X$ and $y \in Y$,

1. $R^\circ(x_1, x_2, x_s)$ holds, if and only if $R(x_1, x_2, y_s)$ does not hold,

2. $R^\rightarrow(x_1, y_t, y)$ holds, if and only if $R(x_1, x_t, y)$ does not hold,

3. $R^\leftarrow(y_t, x_2, y)$ holds, if and only if $R(x_t, x_2, y)$ does not hold,
4. $S^\Diamond(x, x_s)$ holds, if and only if $S(x, y_s)$ does not hold,

5. $S^\Box(y, y)$ holds, if and only if $S(x_1, y)$ does not hold.

Proof. (Item 1). ($\Rightarrow$). Because $R(x_1, x_2, y_s)$ $\iff \forall x' \in X. [R^\circ(x_1, x_2, x') \implies x' \not\leq y_s]$, we claim that there exists $x' \in X$ such that $R^\circ(x_1, x_2, x')$ but $x' \not\leq y_s$, on the assumption that $R^\circ(x_1, x_2, x_s)$. However, this is straightforward, since $(x_s, y_s)$ is a splitting pair, $x_s \not\leq y_s$. (⇐). Assume that $R(x_1, x_2, y_s)$ does not hold. By $\circ$-tightness of Definition 4.2, there exists $x' \in X$ such that $R^\circ(x_1, x_2, x')$ but $x' \not\leq y_s$. As $(x_s, y_s)$ is a splitting pair, $x_s \leq x'$ holds. Furthermore, by the transitivity of $R^\circ$, we obtain $R^\circ(x_1, x_2, x_s)$.

(Item 2). ($\Rightarrow$). Suppose that $R^\leftarrow(x_1, y_1, y)$. Since $(x_1, y_1)$ is a splitting pair, $x_1 \not\leq y$. Hence, by the fact that $R(x_1, x_1, y) \iff \forall y' \in Y. [R^\leftarrow(x_1, y', y) \implies x_1 \leq y']$, we conclude that $R(x_1, x_1, y)$ does not hold. (⇐). Assume that $R(x_1, x_1, y)$ does not hold. By $\leftarrow$-tightness, there exists $y' \in Y$ such that $R^\leftarrow(x_1, y', y)$ and $x_1 \not\leq y'$. Since $(x_1, y_1)$ is a splitting pair, $y' \leq y_1$ holds. By the transitivity of $R^\leftarrow$, we obtain that $R^\leftarrow(x_1, y_1, y)$.

(Item 3). ($\Rightarrow$). Suppose that $R^\leftarrow(y_1, x_2, y)$. Because $(x_1, y_1)$ is a splitting pair, we have $x_1 \not\leq y$. Since $R(x_1, x_1, y) \iff \forall y' \in Y. [R^\leftarrow(y', x_2, y) \implies x_1 \leq y']$, we conclude that $R(x_1, x_2, y)$ does not hold. (⇐). Assume that $R(x_1, x_2, y)$ does not hold. By $\leftarrow$-tightness, there exists $y' \in Y$ such that $R^\leftarrow(y', x_2, y)$ and $x_1 \not\leq y'$. As $(x_1, y_1)$ is a splitting pair, $y' \leq y_1$ holds. By the transitivity of $R^\leftarrow$, we obtain that $R^\leftarrow(y_1, x_2, y)$.

(Item 4). ($\Rightarrow$). Assume that $S^\Diamond(x, x_s)$. Since $(x_s, y_s)$ is a splitting pair, we have $x_s \not\leq y_s$. Hence, by the fact that $S(x, y_s) \iff \forall x' \in X. [S^\Diamond(x, x') \implies x' \not\leq y_s]$, we conclude that $S(x, y_s)$ does not hold. (⇐). Assume that $S(x, y_s)$ does not hold. By $\Diamond$-tightness, there exists $x' \in X$ such that $S^\Diamond(x, x')$ but $x' \not\leq y_s$. Since $(x_s, y_s)$ is a splitting pair, $x_s \leq x'$ holds. So, by the transitivity of $S^\Diamond$, we obtain that $S^\Diamond(x, x_s)$.

(Item 5). ($\Rightarrow$). Assume that $S^\Box(y, y)$. As $(x_s, y_s)$ is a splitting pair, $x_s \not\leq y_s$. Hence, by the fact that $S(x, y_s) \iff \forall y' \in Y. [S^\Box(y', y) \implies x_s \leq y']$, we conclude that $S(x, y_s)$ does not hold. (⇐). Assume that $S(x, y_s)$ does not hold. By $\Box$-tightness, there exists $y' \in Y$ such that $S^\Box(y', y)$ and $x_s \not\leq y'$. Since $(x_s, y_s)$ is a splitting pair, $y' \leq y_s$ holds. So, by the transitivity of $S^\Box$, we obtain that $S^\Box(y, y)$. $\Box$

Proposition 4.12. Let $M$ be a bi-approximation model. For each splitter $x_s \in X_s$ and each splitter $y_t \in Y_t$, we have

1. $M \models \phi \lor \psi \iff M \models \phi$ or $M \models \psi$,
2. $M \models_y \phi \land \psi \iff M \models_{y_1} \phi$ or $M \models_{y_2} \psi$,
3. $M \models_{x_t} \phi \circ \psi \iff \exists x_1, x_2 \in X. [M \models_{x_t} \phi, M \models_{y_1} \psi$ and $R^\circ(x_1, x_2, x_s)]$,
4. $M \models_y \phi \rightarrow \psi \iff \exists x_1 \in X, y \in Y. [M \models_{x_t} \phi, M \models_{y} \psi$ and $R^\rightarrow(x_1, y_1, y)]$,
5. $M, \psi \models \phi \iff \exists x \in X, y \in Y. [M, \models \phi, M, \models \psi \text{ and } R^\pm(y, x, y)]$.

6. $M, \models \phi \iff \exists x \in X. [M, \models \phi \text{ and } S_0(x, x_\Lambda)]$.

7. $M, \models \phi \iff \exists x \in Y. [M, \models \phi \text{ and } S_{\Lambda}(y, y)]$.

**Proof.** Item 1 and item 2 are in [21, Lemma 3.3]. So, here we show the other cases. Also, note that all the $\leftarrow$-directions are trivial, hence we consider the $\Rightarrow$-directions only.

(Item 3). We prove the contraposition. Assume that, for arbitrary $x_1, x_2 \in X$, if $M, \models \phi$ and $M, \models \psi$ then $R^\pm(x_1, x_2, x_1) \neq y$. By Proposition 4.11, $R(x_1, x_2, y)$ holds for a splitting counterpart $y_s$ of $x_s$. Hence, by definition, $M, \models \phi \circ \psi$. Furthermore, since $y_s$ is a splitting counterpart of $x_s$, we also have $x_s \not\leq y_s$. Thus, $M, \not\models \phi \rightarrow \psi$.

(Item 4). We prove the contraposition. Assume that, for arbitrary $x_1 \in X$ and $y \in Y$, if $M, \models \phi$ and $M, \not\models \psi$ then $R^\pm(x_1, y, y) \neq y_s$. By Proposition 4.11, $R(x_1, x_1, y)$ holds for a splitting counterpart $x_t$ of $y_t$. So, by definition, we have $M, \models \phi \rightarrow \psi$. In addition, since $x_t$ is a splitting counterpart of $y_t$, $x_t \not\leq y_t$. Hence, we obtain that $M, \not\models \phi \rightarrow \psi$.

(Item 5). We prove the contraposition. Assume that, for arbitrary $x_2 \in X$ and $y \in Y$, if $M, \models \phi$ and $M, \models \psi$ then $R^\pm(y, x_2, x_2) \neq y$. Due to Proposition 4.11, $R(x_2, x_2, y)$ holds for a splitting counterpart $x_t$ of $y_t$. Hence, by definition, $M, \models \phi \leftarrow \phi$. In addition, by $x_t \not\leq y_t$, we obtain that $M, \models \phi \leftarrow \phi$.

(Item 6). We prove the contraposition. Assume that, for any $x \in X$, if $M, \models \phi$ then $S_0(x, x_s)$ does not hold. So, by Proposition 4.11, $S(x, y_t)$ holds for a splitting counterpart $y_s$ of $x_s$. By definition, $M, \models \phi$. Furthermore, since $y_s$ is a splitting counterpart of $x_s$, we have $x_s \not\leq y_s$, which derives $M, \models \phi$.

(Item 7). We prove the contraposition. Assume that, for any $y \in Y_s$, if $M, \models \phi$ then $S_{\Lambda}(y, y)$ does not hold. Because of Proposition 4.11, we have that $S(x_t, y)$ for a splitting counterpart $x_t$ of $y_t$. Hence, $M, \models \phi$. Additionally, as $x_t$ is a splitting counterpart of $y_t$, we also have that $x_t \not\leq y_t$. Therefore, $M, \not\models \phi$. \hfill \Box

5. **Invariance of validity and first-order definability**

In the previous section, we have seen distributive polarity frames as a possible characterisation of the distributivity on polarity frames and the essential roles of splitters in polarity frames of distributive modal substructural logics. In this section, we shall compare polarity frames of distributive modal substructural
logics with Kripke frames of distributive modal substructural logics, by introducing intermediary structures, named the prime skeleton, and show invariance of validity of sequents among these frames first.

A prime skeleton for distributive polarity frames has been introduced in [21] to investigate how essentially splitters work on distributive polarity frames: for each distributive polarity frame $\langle X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y \rangle$, we call the triple $\langle X_s, Y_s, \leq_s \rangle$ the prime skeleton of $\langle X, Y, \leq \rangle$, where $X_s$ is the set of all splitters in $X$, $Y_s$ the set of all splitters in $Y$, and $\leq_s$ the extended preorder by the restriction $B_s$ of $B$ on $X_s \times Y_s$. In addition, it has proved, with help of [25], invariance of validity of sequents between distributive polarity frames and the prime skeletons. Here we expand the results to polarity frames of distributive modal substructural logics $\mathcal{F} = \langle X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y \rangle$.

For a polarity frame of distributive modal substructural logics $\mathcal{F} = \langle X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y \rangle$, the nonuple $\mathcal{F}_s = \langle X_s, Y_s, \leq_s, R_s, S_s, O_X^s, O_Y^s, N_X^s, N_Y^s \rangle$ is the prime skeleton of $\mathcal{F}$, where $\langle X_s, Y_s, \leq_s \rangle$ is the prime skeleton of $\langle X, Y, \leq \rangle$, and $R_s$, $S_s$, $O_X^s$, $O_Y^s$, $N_X^s$, and $N_Y^s$ are the restrictions of $R$, $S$, $O_X$, $O_Y$, $N_X$, and $N_Y$ on $X_s$ and $Y_s$, respectively.

**Theorem 5.1.** Let $\mathcal{F} = \langle X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y \rangle$ be a polarity frame of distributive modal substructural logics. The prime skeleton $\mathcal{F}_s = \langle X_s, Y_s, \leq_s, R_s, S_s, O_X^s, O_Y^s, N_X^s, N_Y^s \rangle$ is also a polarity frame of distributive modal substructural logics. Furthermore, the underlying polarity frame of $\mathcal{F}_s$ is prime.

**Proof.** Firstly, we show that $\langle X_s, Y_s, \leq_s \rangle$ is a prime polarity frame. Since $\langle X, Y, \leq \rangle$ is a non-trivial distributive polarity frame by definition, there exist $x \in X$ and $y \in Y$ such that $x \not\leq y$. By the splitting condition, there must exist a splitting pair $(x_s, y_s)$ such that $x_s \leq x$ and $y \leq y_s$. Also, as $(x_s, y_s)$ is a splitting pair, $x_s \not\leq y_s$. Hence, $X_s$ and $Y_s$ are non-empty, which concludes that $\langle X_s, Y_s, \leq_s \rangle$ is a prime polarity frame. We also mention that the non-triviality of $\langle X_s, Y_s, \leq_s \rangle$ follows from $x_s \not\leq y_s$.

Next we prove that the non-emptiness of $O_X^s$ and $O_Y^s \neq Y_s$. Because $O_Y$ is a proper subset of $Y$, there exists $y \in Y$ such that $y \not\in O_Y$. By Proposition 4.6, it means that there exists $x \in O_X$ such that $x \not\leq y$. Due to the splitting condition, there exists a splitting pair $(x_s, y_s)$ such that $x_s \leq x$ and $y_s \leq y$. As $O_X$ is downward closed, we have $x_s \in O_X$, hence $x_s \in O_X^s$. Furthermore, since $x_s \not\leq y_s$, we also obtain that $y_s \not\in O_Y^s$ by Proposition 4.6. Therefore, $O_X^s$ is non-empty and $O_Y^s \neq Y_s$.

Next we consider the tightness conditions. This is in fact straightforward from Proposition 11. Here we check the $\rightarrow$-tightness only. We prove the contraposition. For arbitrary $x_i, x_j \in X_s$ and $y_i \in Y_s$, if $R(x_i, x_j, y_i$) does not hold, with help of Proposition 11, we have that $R^{-1}(x_i, y_i, y_k)$. In addition, $y_j$ is a splitting counterpart of $x_j$, we also have that $x_j \not\leq y_j$. Therefore, $\rightarrow$-tightness holds. The other cases are just analogous.

Next, we show the $R$-conditions and the $S$-transitivity. However, since the domains are subsets of $X$ and $Y$, $R$-transitivity and $S$-transitivity hold trivially. Hence the real problem is to show the existence of appropriate splitters for $R$-order, $R$-identity and $R$-associativity.
(R-order). For arbitrary \( x_s, x_t \in X_s \), if \( x_t \leq_s x_s \), by R-order on \( F \), there exists \( o \in O_X \) such that \( R^o(x_s, o, x_t) \) or \( R^o(o, x_s, x_t) \). Assume that \( R^o(x_s, o, x_t) \) holds. Then we claim that \( R(x_s, o, y_t) \) does not hold. This fact can be proved by the analogous method to Proposition 4.11. That is, \( R^o(x_s, o, x_t) \) but \( x_t \not\leq y_t \), hence \( R(x_s, o, y_t) \) does not hold. Now, we apply \( \to \)-tightness on \( F \) for \( x_s, o \) and \( y_t \), and obtain that there exists \( y \in Y \) such that \( R^o(x, y, y_t) \) but \( o \not\leq y \). By the splitting condition, there exists a splitting pair \((o_s, y_0)\) such that \( o_s \leq o \) and \( y \leq y_0 \). Since \( O_X \) is downward closed, \( o_s \in O_X \), hence \( o_s \in O^*_X \). Moreover, by the transitivity of \( R^{-1} \), we also have that \( R^{-1}(x_s, y_0, y_t) \). Hence, by Proposition 4.11, we conclude that \( R^o(x_s, o, x_t) \). We can analogously prove the case that \( R^o(o, x_s, x_t) \) holds.

(R-identity). For any \( x_s \in X_s \), by R-identity on \( F \), there exists \( o \in O_X \) such that \( R^o(o, x_s, x_s) \). Since \( R^o(o, x_s, y_s) \) does not hold, by applying \( \leftrightarrow \)-tightness on \( F \) for \( o, x_s \) and \( y_s \), we obtain an element \( y \in Y \) satisfying \( R^o(y, x_s, y_s) \) and \( o \not\leq y \). By the splitting condition, there exists a splitting pair \((o_s, y_0)\) such that \( o_s \leq o \) and \( y \leq y_0 \). As \( O_X \) is downward closed, \( o_s \in O_X \), hence \( o_s \in O^*_X \). Moreover, by the transitivity of \( R^{-1} \), we also gain that \( R^{-1}(y_0, x_s, y_s) \). Therefore, by Proposition 4.11, we derive \( R^o(o, x_s, x_s) \). The other side is also analogous.

(R-associativity). For arbitrary \( x_s, x_t, x_u, x_v \in X_s \), if there exists \( x_s \in X_s \) such that \( R^o(x_s, x_t, x_u) \) and \( R^o(x_t, x_u, x_v) \), by R-associativity on \( F \), there exists \( x \in X \) such that \( R^o(x, x_u, x_v) \) and \( R^o(x, x_t, x_v) \). By \( R^o(x, x_u, x_v) \), we state that \( R(x, x_u, y_u) \) does not hold. Then we apply \( \leftrightarrow \)-tightness on \( F \) for \( x, x_u \) and \( y_u \), and get an element \( y \in Y \) satisfying \( R^o(y, x_u, y_u) \) and \( x \not\leq y \). As \( F \) is distributive, there exists a splitting pair \((x_j, y_j)\) such that \( x_j \leq x \) and \( y \leq y_j \). Here, by the transitivity of \( R^{-1} \), we obtain that \( R^{-1}(y_j, x_u, y_u) \). Plus, by Proposition 4.11, \( R^o(x_j, x_u, x_v) \). Additionally, by the transitivity of \( R^o \) and our assumption \( R^o(x_s, x_t, x) \), we also have that \( R^o(x_s, x_t, x_j) \). The converse direction is just analogous. Therefore \( F_s \) is a polarity frame of distributive modal substructural logics. \( \Box \)

Next we compare validity of sequents on the prime skeletons with that on p-frames via Dedekind-cut preserving morphisms introduced in [25].

**Definition 5.2** (Dedekind-cut preserving morphism). Let \( F = \langle X_1, Y_1, \leq_1, R_1, S_1, O_{X_1}, O_{Y_1}, N_{X_1}, N_{Y_1} \rangle \) and \( G = \langle X_2, Y_2, \leq_2, R_2, S_2, O_{X_2}, O_{Y_2}, N_{X_2}, N_{Y_2} \rangle \) be p-frames. A pair \((\sigma, \tau)\) of functions \( \sigma : X_1 \to X_2 \) and \( \tau : Y_1 \to Y_2 \) is a Dedekind-cut preserving morphism from \( F \) to \( G \), a d-morphism for short, if

1. if \( \sigma(x) \leq_2 \tau(y) \) then \( x \leq_1 y \),
2. if \( \forall z \in Y_1. [y' \leq_2 \tau(z) \implies x \leq_1 z] \) then \( \sigma(x) \leq_2 y' \),
3. if \( \forall z \in X_1. [\sigma(z) \leq_2 x' \implies z \leq_1 y] \) then \( x' \leq_2 \tau(y) \),
4. if \( R_2(\sigma(x_1), \sigma(x_2), \tau(y)) \) then \( R_1(x_1, x_2, y) \),
5. if \( \forall z_1, z_2 \in X_1. [\sigma(z_1) \leq_2 x_1' \text{ and } \sigma(z_2) \leq_2 x_2' \implies R_1(z_1, z_2, y)] \) then \( R_2(x_1', x_2', \tau(y)) \).

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6. if \( \forall z_1 \in X_1, z \in Y_1, [\sigma(z_1) \leq_2 x'_1 \text{ and } y' \leq_2 \tau(z) \implies R_1(z_1, x_2, z)] \) then 
   \( R_2(x'_1, \sigma(x_2), y') \),
7. if \( \forall z_2 \in X_1, z \in Y_1, [\sigma(z_2) \leq_2 x'_2 \text{ and } y' \leq_2 \tau(z) \implies R_1(x_1, z_2, z)] \) then 
   \( R_2(\sigma(x_1), x'_2, y') \),
8. if \( S_2(\sigma(x), \tau(y)) \) then \( S_1(x, y) \),
9. if \( \forall z \in X_1, [\sigma(z) \leq_2 x' \implies S_1(z, y)] \) then \( S_2(x', \tau(y)) \),
10. if \( \forall z \in Y_1, [y' \leq_2 \tau(z) \implies S_1(x, z)] \) then \( S_2(\sigma(x), y') \),
11. \( [x \in O_{X_1} \iff \sigma(x) \in O_{X_2}] \) and \( [x \in N_{X_1} \iff \sigma(x) \in N_{X_2}] \),
12. \( [y \in O_{Y_1} \iff \tau(y) \in O_{Y_2}] \) and \( [y \in N_{Y_1} \iff \tau(y) \in N_{Y_2}] \).

In addition, if a bounded morphism \( \langle \sigma \rangle \) satisfies

13. if \( x \leq_1 y \) then \( \sigma(x) \leq_2 \tau(y) \),
14. if \( \forall z \in X_1, z \in Y_1, [\sigma(z_x) \leq_2 x' \text{ and } y' \leq_2 \tau(z_y) \implies z_x \leq_1 z_y] \) then 
   \( x' \leq_2 y' \),

we call \( \langle \sigma \rangle \) \( B \)-describing. In each case, all (decorated) \( x \) and \( y \) are universally quantified, and all (decorated) \( x, x', y \) and \( y' \) range over \( X_1, X_2, Y_1 \) and \( Y_2 \), respectively.

Let \( F = (X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y) \) be a p-frame and \( F_s = (X_s, Y_s, \leq_s, R_s, S_s, O_{X_s}, O_{Y_s}, N_{X_s}, N_{Y_s}) \) the prime skeleton of \( F \). Since the domains \( X_s \) and \( Y_s \) of the prime skeleton are subsets of \( X \) and \( Y \), there are the natural embedding functions \( \epsilon_X : X_s \to X \) and \( \epsilon_Y : Y_s \to Y \). In fact, the pair \( \langle \epsilon_X, \epsilon_Y \rangle \) is a \( B \)-describing bounded morphism from \( F_s \) to \( F \). Therefore, by the fact [25, Theorem 4.8], we obtain the following.

**Theorem 5.3.** Let \( F \) be a p-frame and \( F_s \) the prime skeleton. For each sequent \( \phi \Rightarrow \psi \), we have

\[ F \models \phi \Rightarrow \psi \iff F_s \models \phi \Rightarrow \psi. \]

Next we compare prime p-frames, namely p-frames whose underlying polarity frames are prime, with Kripke frames.

Let \( F = (X, Y, \leq, R, S, O_X, O_Y, N_X, N_Y) \) be a prime p-frame. We let the sextuple \( \text{isol}(F) = (X, \leq, R^2, S^0, O_X, N_X) \) be the isolation of \( F \), where \( R^2(x_1, x_2, x) \) and \( S^0(x, x') \) are the abbreviations given before, namely \( \forall y \in Y, [R(x_1, x_2, y) \implies x \leq y] \) and \( \forall y \in Y, [S(x, y) \implies x' \leq y] \). It is straightforward to show that the isolation of \( F \), \( \text{isol}(F) \) is a Kripke frame of distributive modal substructural logics. Furthermore, we can prove invariance of validity of sequents (formulae).

**Proposition 5.4.** Let \( F \) be a prime p-frame and \( \text{isol}(F) \) the isolation of \( F \). For every formula \( \phi \), we have

\[ F \models t \Rightarrow \phi \iff \text{isol}(F) \models t \Rightarrow \phi. \]

Remember that every sequent \( \phi \Rightarrow \psi \) can be seen as \( t \Rightarrow \phi \to \psi \) on substructural logic.

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Proof. First of all, for every doppelgänger valuation $V$, $V^\downarrow$ is a valuation on $\text{isol}(F)$. Also, every valuation $V_K$ on $\text{isol}(F)$ generates a doppelgänger valuation on $F$. The former statement is rather trivial, because $V^\downarrow$ is a map from each propositional variable $p$ to a downset of $X$. For the latter one, we define a pair $(V_K^\downarrow, V_K^\uparrow)$ of functions $V_K^\downarrow : \Phi \rightarrow \wp(X)$ and $V_K^\uparrow : \Phi \rightarrow \wp(Y)$ as follows: for every $p \in \Phi$, we let

1. $V_K^\downarrow(p) := V_K(p)$,
2. $V_K^\uparrow(p) := \{y \in Y \mid \forall x \in V_K(p). x \leq y\}$.

We also need to check that the pair $(V_K^\downarrow, V_K^\uparrow)$ forms a doppelgänger valuation. The only non-trivial condition is that $V_K^\downarrow(p) \supseteq \{x \in X \mid \forall y \in V_K(p). x \leq y\}$. We prove the contraposition.

Suppose that $x_s \notin V_K(p)$. Since $F$ is prime, there exists a splitting counterpart $y_s \in Y$ of $x_s$, hence $x_s \not\leq y_s$. We show that $y_s \in V_K(p)$. For any $x \in V_K(p)$, if $x \not\leq y_s$ then $x \in V_K$, because $x_s \leq x$ and $V_K(p)$ is downward closed. It contradicts to the assumption $x_s \notin V_K(p)$, hence $x \leq y_s$ for all $x \in V_K(p)$. Thus we obtain $y_s \in V_K(p)$. Therefore, there is one-to-one correspondence between doppelgänger valuations $V$ on $F$ and valuations $V_K$ on $\text{isol}(F)$, which satisfies $x \in V^\downarrow(p) \iff x \in V_K(p)$ for every propositional variable $p \in \Phi$.

Next we claim that $(F, V) \models \phi \iff (\text{isol}(F), V_K), x_s \models \phi$, where $V$ and $V_K$ are the corresponding valuations. We prove it by induction. The base cases follow from $V^\downarrow(p) = V_K(p)$. The inductive steps are from Proposition 4.12.

Finally, we show that $F \models t \Rightarrow \phi \iff \text{isol}(F) \models t \phi. (\Rightarrow)$. Contraposition. Suppose that $\text{isol}(F) \not\models t \phi$, that is, there exist a valuation $V_K$ on $\text{isol}(F)$ and $x_s \in O_X$ such that $(\text{isol}(F), V_K), x_s \not\models \phi$. As we saw above, we have that $(F, V) \not\models \phi \iff (\text{isol}(F), V_K), x_s \models \phi$, where $V$ and $V_K$ are the corresponding valuations. Hence, for the corresponding doppelgänger valuation $V$ to the valuation $V_K$, we obtain that $(F, V) \not\models \phi$. Due to Theorem 4.7, $(F, V) \not\models \phi$ for a splitting counterpart $y_s$ of $x_s$. In addition, since $x_s \in O_X$, we also have that $(F, V) \not\models s$.

Thus, because of $x_s \not\leq y_s$, we conclude that $(F, V) \not\models s \Rightarrow \phi$, hence $F \not\models t \Rightarrow \phi$.

$(\Leftarrow)$. Contraposition. Suppose that $F \not\models t \Rightarrow \phi$, that is, there exist a doppelgänger valuation $V$, $x_s \in X$ and $y_t \in Y$ such that $(F, V) \not\models s$ and $(F, V) \not\models t$ but $x_s \not\leq y_t$. Since $x_s$ is a splitter, hence there is a splitting counterpart $y_s$ of $x_s$, which satisfies $y_t \leq y_s$. Now, by the Hereditary (Proposition 4.5), we obtain that $(F, V) \models s \phi$. However, as $x_s \notin y_s$, we derive $(F, V) \not\models \phi$ by Proposition 4.6. For the corresponding valuation $V_K$ to $V$, we obtain that $(\text{isol}(F), V_K), x_s \models t$ and $(\text{isol}(F), V_K), y_s \not\models \phi$. Therefore, $\text{isol}(F) \not\models t \phi$.

So far we have studied the isolation of polarity frames of distributive modal substructural logics and invariance of validity of sequents. Next, conversely,
we construct polarity frames, called the polarisation, from Kripke frames of distributive modal substructural logics.

Let $K = (W, \preceq, R_\circ, S_\circ, O, N)$ be a Kripke frame of distributive modal substructural logics. We let $W_X$ and $W_Y$ be the isomorphic copy of $W$. To distinguish elements of $W_X$ and $W_Y$ from $W$, we denote elements of $W_X$ by $x_i, x_j, \ldots$ and elements of $W_Y$ by $y_i, y_j, \ldots$. In addition, for the corresponding elements we add the same index, that is, $w_i$ corresponds to $x_i$ and $y_i$. We introduce a binary relation $\preceq$ on $W_X \times W_Y$ as follows: for each $x_i \in W_X$ and each $y_j \in W_Y$, $x_i \preceq y_j \iff w_i \not\preceq w_j$. Then we can see the triple $\langle W_X, W_Y, \preceq \rangle$ as a polarity frame. We can also show that $w_i \preceq w_j \iff x_i \preceq x_j \iff y_i \preceq y_j$.

Here we prove the first equivalence only. ($\Rightarrow$). Assume that $w_i \preceq w_j$. For any $y_k \in W_Y$, if $x_j \preceq y_k$, by definition, $w_k \not\preceq w_j$. Thus we obtain that $w_k \not\preceq w_i$, it contradicts otherwise. Again, by the definition of $\preceq$, we conclude $x_i \preceq y_k$, hence $x_i \preceq x_j$. ($\Leftarrow$). Contraposition. Suppose that $w_i \not\preceq w_j$. By definition, we have $x_j \not\preceq y_k$. However, since $\preceq$ is reflexive, we have $w_i \preceq w_i$, so $x_i \preceq y_i$. Therefore, $x_i \preceq x_j$.

**Proposition 5.5.** The polarity frame $\langle W_X, W_Y, \preceq \rangle$ is prime. Furthermore, the corresponding elements $(x_s, y_s)$ form a splitting pair.

**Proof.** Since $W_X$ and $W_Y$ are isomorphic copies of $W$, every element in $W_X$ and in $W_Y$ has the corresponding element in $W_Y$ and in $W_X$. Hence, it suffices to show that each pair $(x_s, y_s)$ forms a splitting pair.

As $\preceq$ is reflexive, $w_s \preceq w_s$, which derives $x_s \not\preceq y_s$ by definition. For any $y_k \in W_Y$, if $x_s \not\preceq y_k$ then $w_i \preceq w_k$. In addition, because of $w_i \preceq w_s \iff y_k \preceq y_s$, we conclude $y_k \preceq y_s$. Thus $(x_s, y_s)$ forms a splitting pair.

On top of the prime polarity frame $\langle W_X, W_Y, \preceq \rangle$, we also define a ternary relation $R'$ on $W_X \times W_X \times W_Y$, a binary relation $S'$ on $W_X \times W_Y$, subsets $O^x$, $O^n$, $N^x$ and $N^y$ as follows:

1. for all $x_i, x_j \in W_X$ and $y_k \in W_Y$, $R'(x_i, x_j, y_k)$ holds, if and only if $R_\circ(w_i, w_j, w_k)$ does not hold,
2. for all $x_i \in W_X$ and $y_j \in W_Y$, $S'(x_i, y_j)$ holds, if and only if $S_\circ(w_i, w_j)$ does not hold,
3. $O^x := \{x_s \in W_X \mid w_s \in O\}$ and $N^x := \{x_s \in W_X \mid w_s \in N\}$,
4. $O^n := \{y_t \in W_Y \mid w_t \not\in O\}$ and $N^y := \{y_t \in W_Y \mid w_t \not\in N\}$.

**Theorem 5.6.** Let $K$ be a Kripke frame of distributive modal substructural logics. The nonuple $\text{pol}(K) = \langle W_X, W_Y, \preceq, R', S', O^x, O^n, N^x, N^y \rangle$ is a prime polarity frame of distributive modal substructural logics.

**Proof.** We insist that $R'^x(x_i, x_j, x_k) \iff R_\circ(w_i, w_j, w_k)$ for all $x_i, x_j, x_k \in W_X$ and $w_i, w_j, w_k \in W$. Then, $R$-order, $R$-identity and $R$-associativity are rather trivial. Hence, we show the equivalence, first.
Contraposition. Suppose that $R_o(w_i, w_j, w_k)$ does not hold. By the definition of $R'$, we obtain that $R'(x_i, x_j, y_k)$. Further, since $x_k \not\leq y_k$, we conclude that $R''(x_i, x_j, x_k)$ does not hold. ($\Leftarrow$). Assume that $R_o(w_i, w_j, w_k)$. For any $y_i \in W_Y$, if $R'(x_i, x_j, y_i)$ holds, as we saw above, $R_o(w_i, w_j, w_k)$ does not hold. Hence, $w_i \not\leq w_k$, it contradicts otherwise. By the definition of $\leq$, we conclude $x_k \leq y_i$, which means $R'''(x_i, x_j, x_k)$. Notice that we can analogously prove

1. $R''^-(x_i, y_j, y_k) \iff R_o(w_i, w_j, w_k)$,
2. $R''^-(y_i, x_j, y_k) \iff R_o(w_i, w_j, w_k)$,
3. $S^{\Diamond}(x_i, x_j) \iff S_o(w_i, w_j)$,
4. $S^{\Box}(y_i, y_j) \iff S_o(w_i, w_j)$,

for all $x_i, x_j \in W_X, y_i, y_j, y_k \in W_Y$ and $w_i, w_j, w_k \in W$. $R$-transitivity and $S$-transitivity are nothing but the contraposition of $R_o$-transitivity and $S_o$-transitivity, hence they also hold.

(O-isom). For arbitrary $x_s \in O^x$ and $y_t \in O^y$, by definition, we have that $w_s \in O$ and $w_t \not\in O$. Hence, $w_t \not\leq w_s$, it contradicts otherwise. So, by the definition of $\leq$, we obtain that $x_s \leq y_t$. Conversely, for any $x_s \not\in O^x$, by definition, we have $w_s \not\in O$, hence $y_s \in O^y$. As $x_s \not\leq y_s$, we conclude that $x_s \not\in \{x \in W_X | \forall y \in O^y, x \leq y\}$. The other case of O-isom and N-isom are just analogous. We also mention that $O^x$ is non-empty because $O$ is non-empty. In addition, $O^y$ is a proper subset of $W_Y$, because the splitting counterparts of elements in $O^x$ cannot be in $O^y$ (and $O^x$ is non-empty).

In the rest, we show the tightness conditions. Here we show the contraposition. ($\Leftarrow$-tightness). For arbitrary $x_i, x_j \in W_X$ and $y_k \in W_Y$, if $R'(x_i, x_j, y_k)$ does not hold, $R_o(w_i, w_j, w_k)$ holds by definition. As we saw above, we have $R''^o(x_i, x_j, x_k)$. Furthermore, $x_k \not\leq y_k$. ($\implies$-tightness). For arbitrary $x_i, x_j \in W_X$ and $y_k \in W_Y$, if $R'(x_i, x_j, y_k)$ does not hold, $R_o(w_i, w_j, w_k)$ holds by definition. As the above equivalence, we have $R''^o(x_i, y_j, y_k)$, and also $x_j \not\leq y_j$. ($\Leftarrow$-tightness). For arbitrary $x_i, x_j \in W_X$ and $y_k \in W_Y$, if $R'(x_i, x_j, y_k)$ does not hold, $R_o(w_i, w_j, w_k)$ holds by definition. As the above equivalence, we have $R''^o(y_i, x_j, y_k)$, and also $x_j \not\leq y_j$. ($\Diamond$-tightness). For arbitrary $x_i \in W_X$ and $y_j \in W_Y$, if $S'(x_i, y_j)$ does not hold, by definition, $S_o(w_i, w_j)$ holds. As the above equivalence, we have $S^{\Diamond}(y_i, y_j)$ and also $x_i \not\leq y_i$. Thus all the tightness conditions hold.

Next we consider invariance of validity of sequents between Kripke frames and the polarisation. Let $K$ and $pol(K)$ be a Kripke frame and the polarisation of $K$. For any valuation $V_K$ on $K$, we construct a doppelganger valuation $V$ as follows: for any $p \in \Phi$, we let

1. $V^+(p) := V_K(p)$,
2. $V_1(p) := \{y \in W_Y \mid w_t \not\in V_K(p)\}$.

Let us check that these form a doppelgänger valuation: $V^1(p) = \{x \in W_X \mid \forall y \in V_1(p), x \leq y\}$ and $V_1(p) = \{y \in W_Y, \forall x \in V^1(p). x \leq y\}$.

For arbitrary $x_s \in V^1(p)$ and $y_t \in V_1(p)$, by definition, we have that $w_s \in V_K(p)$ and $w_t \not\in V_K(p)$. As $V_K(p)$ is downward closed, $w_t \not\in w_s$, it contradicts otherwise. Hence, $x_s \leq y_t$. Conversely, suppose that $x_s \not\in V^1(p)$. By definition, $w_s \not\in V_K(p)$. Thus, $y_s \in V_1(p)$. Because of $x_s \not\leq y_s$, which means that $x_s \not\in \{x \in W_X \mid \forall y \in V_1(p), x \leq y\}$. For arbitrary $y_t \in V_1(p)$ and $x_s \in \bar{V}^1(p)$, by definition, we have that $y_t \not\in V_K(p)$ and $w_s \not\in V_K(p)$. Since $V_K(p)$ is downward closed, $w_t \not\in w_s$, it contradicts otherwise. Hence, $x_s \leq y_t$. Conversely, suppose that $y_t \not\in V_1(p)$. By definition, we have that $w_t \not\in V_K(p)$, which means that $x_t \not\in V^1(p)$ as well. Because of $x_t \not\leq y_t$, we conclude that $y_t \not\in \{y \in W_Y \mid \forall x \in V^1(p), x \leq y\}$. Therefore, they form a doppelgänger valuation. Furthermore, it is trivial to show that $V^1$ corresponds to $V_K$, hence there is one-to-one correspondence between valuations $V_K$ on $\mathbb{K}$ and doppelgänger valuations $V$ on the polarisation $\text{pol}(\mathbb{K})$.

**Theorem 5.7.** Let $\mathbb{K}$ be a Kripke frame and $\text{pol}(\mathbb{K})$ the polarisation of $\mathbb{K}$. For every formula $\phi$, we have

$$\mathbb{K} \vDash \phi \iff \text{pol}(\mathbb{K}) \vDash \text{pol}(\mathbb{K}) \vDash \phi.$$

**Proof.** We claim that $(\mathbb{K}, V_K)$, $w_s \vDash \phi \iff (\text{pol}(\mathbb{K}), V) \vDash \phi$ for the corresponding valuations $V_K$ and $V$. The base cases follow from $V_K(p) = V^1(p)$ for every propositional variable $p \in \Phi$. And, the inductive steps are straightforward from Proposition 4.12.

($\Rightarrow$). We prove the contraposition. Suppose that $\mathbb{K} \not\vDash \text{pol}(\mathbb{K}) \vDash \phi$, that is, there exist a doppelgänger valuation $V$ on $\text{pol}(\mathbb{K})$, $x_s \in W_X$ and $y_t \in W_Y$ such that $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$ but $x_s \not\leq y_t$. By definition, we have that $x_s \in O^\phi$, that is, $w_s \in O$. Also, $w_t \preceq w_s$ because of $x_s \not\leq y_t$. Moreover, due to Theorem 4.7, we obtain that $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$ as well. Thus, $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$, it contradicts otherwise. Therefore, by the above equivalence, $\langle \mathbb{K}, V_K \rangle$, $w_s \vDash \phi$ but $\langle \mathbb{K}, V_K \rangle$, $w_s \not\vDash \phi$, namely $\langle \mathbb{K}, V_K \rangle$, $w_s \not\vDash \phi$, hence $\mathbb{K} \not\vDash \phi$.

($\Leftarrow$). We prove the contraposition. Suppose that $\mathbb{K} \not\vDash \phi$, that is, there exist a valuation $V_K$ on $\mathbb{K}$ and $w_s \in O$ such that $\langle \mathbb{K}, V_K \rangle$, $w_s \not\vDash \phi$. By definition, we have $x_s \in O^\phi$, hence $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$ for the corresponding doppelgänger valuation $V$ to $V_K$. Also, by the above equivalence, $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$. Furthermore, by Theorem 4.7, we obtain that $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$. However, since $x_s \not\leq y_s$, we conclude $\langle \text{pol}(\mathbb{K}), V \rangle \vDash \phi$. Therefore, $\mathbb{K} \not\vDash \phi$. □

We have investigated invariance of validity of sequents among Kripke frames, prime polarity frames and polarity frames in distributive modal substructural logics, so far. As a consequence, we state that the soundness and the completeness of DFLM and the extensions with respect to Kripke frames indirectly,

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with help of Theorem 4.8 and Theorem 4.9. Note that the soundness and the completeness are also provable directly as normal.

Next we shall look into the first-order definability on Kripke frames of distributive modal substructural logics. Here, we recall the consistent variable occurrence intuitively, see [23] for the formal characterisation.

Let \( \phi \Rightarrow \psi \) be a sequent. We expect that \( \phi \) is of type \( \phi_\Delta \) and \( \psi \) is of type \( \phi_\Lambda \), defined by the following Backus-Naur forms:

\[
\begin{align*}
\phi_\Delta & ::= \phi_\psi \lor \phi_\psi \mid \phi_\psi \land \phi_\psi \mid \phi_\psi \circ \phi_\psi \mid \phi_\psi \mid \phi_\Lambda, \quad (1) \\
\phi_\Lambda & ::= \phi_\psi \lor \phi_\gamma \mid \phi_\psi \lor \phi_\gamma \mid \phi_\Delta \rightarrow \phi_\gamma \mid \phi_\gamma \leftrightarrow \phi_\psi \mid \boxdot \phi_\gamma \mid \phi_\psi, \quad (2) \\
\phi_\psi & ::= p \mid t \mid f \mid \phi_\psi \lor \phi_\psi \mid \phi_\psi \land \phi_\psi \mid C \land \phi_\psi \mid \phi_\psi \circ C \mid C \circ \phi_\psi \mid \phi_\psi, \quad (3) \\
\phi_\Lambda & ::= p \mid t \mid f \mid \phi_\Lambda \lor C \mid C \lor \phi_\Lambda \mid \phi_\Lambda \land \phi_\Lambda \mid \phi_\Lambda \rightarrow C \mid C \rightarrow \phi_\Lambda \mid \phi_\Lambda \leftrightarrow C \mid C \leftarrow \phi_\psi \mid \boxdot \phi_\Lambda. \quad (4)
\end{align*}
\]

where \( C \) is a constant formula (a well-formed formula contains no propositional variable). We note that the above Backus-Naur forms are for distributive modal substructural logics, which means that we need to consider the appropriate Backus-Naur forms when we treat other logics.

If \( \phi \) is really of type \( \phi_\Delta \) and \( \psi \) is really of type \( \phi_\Lambda \), the sequent \( \phi \Rightarrow \psi \) has consistent variable occurrence. Otherwise, there have to be subformulæ of \( \phi \) and/or \( \psi \) which do not allow us to trace back the above Backus-Naur forms (1) and (2). We call these subformulæ critical. Remember that, when we think about subformulæ, we do not count the Backus-Naur forms (3) and (4), see for example item 10 of Example 5.8. The sequent \( \phi \Rightarrow \psi \) has consistent variable occurrence, if all occurrences of each propositional variables are uniformly signed 'only' under critical subformulæ. Note that the signs of \( \phi \) is opposite to those of \( \psi \), because a sequent \( \phi \Rightarrow \psi \) is the same as a formula \( \phi \rightarrow \psi \) in which positive propositional variables in \( \phi \) are negative and negative propositional variables in \( \phi \) are positive, in distributive modal substructural logics. Notice that, if \( \phi \) is of type \( \phi_\Delta \) and \( \psi \) is of type \( \phi_\Lambda \), then, since there is no critical subformula, the sequent \( \phi \Rightarrow \psi \) has trivially consistent variable occurrence. We also introduce this notion for formulae: a formula \( \phi \) has consistent variable occurrence, if so does the sequent \( t \Rightarrow \phi \). Due to Proposition 2.1, for any sequent \( \phi \Rightarrow \psi \) with consistent variable occurrence, so do the formulæ \( \phi \rightarrow \psi \) and \( \psi \leftarrow \phi \).

**Example 5.8.** The following are examples of sequents which have consistent variable occurrence.

1. \( p \Rightarrow t \) and \( f \Rightarrow p \) (weakening): there is no critical subformula.
2. \( p \Rightarrow p \circ p \) (contraction): the only critical subformula is \( p \circ p \).
3. \( p \circ q \Rightarrow q \circ p \) (exchange): the only critical subformula is \( q \circ p \).
4. \( p \Rightarrow p \rightarrow p \) (mingle): there is no critical subformula.
5. \( p_1 \rightarrow p_2 \Rightarrow (p_1 \rightarrow p_2) \rightarrow (p_1 \rightarrow p_2) \) (weak mingle): the critical subformulae are \( p_1 \rightarrow p_2 \) in the left formula and the assumption \( p_1 \rightarrow p_2 \) of the right formula.
6. \((p_1 \rightarrow p_2) \rightarrow p_1 \Rightarrow p_1\) (Peirce’s law): the only critical subformula is 
\((p_1 \rightarrow p_2) \rightarrow p_1\).

7. \(\Diamond (p \circ q) \land (t \rightarrow p) \Rightarrow \Box (q \land t) \rightarrow \Diamond p\): there is no critical subformula.

8. \(\Box p \rightarrow \Diamond q \Rightarrow \Box (p \rightarrow q)\): the critical subformula is \(\Box p \rightarrow \Diamond q\) only.

9. \(\Box \Diamond p \circ \Diamond q \Rightarrow \Box \Diamond p \Leftrightarrow \Box \Diamond q\): the critical subformulae are \(\Box \Diamond p\) in the left 
formula and \(\Box \Diamond q\).

10. \(\Box (\Diamond p \lor (q \circ p)) \Rightarrow (q \rightarrow p) \lor \Diamond q\): the critical subformula is \(\Box (\Diamond p \lor (q \circ p))\)
only. Note that neither \(\Diamond p\) nor \(q \circ p\) are critical, because we do not trace 
back within the Backus-Naur forms (3) and (4) when we consider critical 
subformulae.

The Sahlqvist theorem [26] shows that every sequent with consistent vari-
able occurrence has first-order correspondents computed by the Sahlqvist-van
Benthem algorithm on bi-approximation semantics.

Example 5.9. By means of the Sahlqvist-van Benthem algorithm, we can ob-
tain the first-order correspondents for the sequents given in Example 5.8. Note 
that the following sentences also simplified “by hand”: see [26] for the details.

Also, notice that in this paper, we are only thinking distributive modal 
substructural logics, hence the conditions of disjunctions and conjunctions in [26]
have to be revised for the following results.

1. \(\forall x. O^y(x) \land \forall y. N^y(y)\).

2. \(\forall x. R^o(x, x, x)\).

3. \(\forall x, x_1, x_2. [R^o(x_1, x_2, x) \Rightarrow R^o(x_2, x_1, x)]\).

4. \(\forall x, y, y_1. [x_1 \leq y_1 \text{ and } R^{-}(x_1, y, y_1) \Rightarrow y_1 \leq y]\).

5. \(\forall x, x_1, x_2, y, y_1, y_2. [R_1 \Rightarrow x \leq y]\),
   where \(R_1\) is \(R(x_2, x, y_2), R(x_2, x_1, y_2), R^{-}(x_2, y_1, y_2)\) and \(R^{-}(x_1, y, y_1)\).

6. \(\forall x, y. [\mathcal{C}_1 \Rightarrow x \leq y]\),
   where \(\mathcal{C}_1\) is \(\forall x_1, x_2. [x_2 \leq y \Rightarrow R(x_2, x_1, y_1)] \Rightarrow R(x_1, x, y)]\)

7. \(\forall x, x_1, x_2, x_3, y_1, y_2. [\mathcal{C}_2, R^o(x_2, x_3, x_1) \text{ and } S^o(x_1, x) \Rightarrow R(x_4, x, y_1)]\),
   where \(\mathcal{C}_2\) is conjunction of 
   \(\text{(a) } S(x_2, y_1),\)
   \(\text{(b) } \forall x', x'' . [O^y(x') \text{ and } R^o(x', x_1, x'' ) \Rightarrow S(x'', y_1)],\)
   \(\text{(c) } \forall x'. [S^o(x_4, x') \Rightarrow N^y(x')].\)

8. \(\forall x, x_1, y_1, y_2. [\mathcal{C}_3 \text{ and } R^{-}(x_1, y_1, y_2) \Rightarrow S(x, y_1)],\)
   where \(\mathcal{C}_3\) is \(\forall x', y'. [\mathcal{C}_3(a) \text{ and } \mathcal{C}_3(b) \Rightarrow R(x', x, y')\] with 
   \(\text{(a) } \mathcal{C}_3(a): \forall y''. [x_1 \leq y'' \Rightarrow S(x', y'')],\)
   \(\text{(b) } \mathcal{C}_3(b): [\mathcal{C}_3(a) \text{ and } \mathcal{C}_3(b) \Rightarrow R(x', x, y')].\)
(b) \( \mathcal{CA}_{(b)}: \forall x''.[x'' \leq y' \implies S(x'', y')] \).

9. \( \forall x, x_1, x_2, x_3, x_4, y_1, y_2. [\mathcal{CA}_4 \text{ and } \mathcal{R}_2 \implies R(x, x_4, y_1)] \),

where \( \mathcal{CA}_4 \) is \( \forall y'. [\forall x'. [S(x', y_2) \implies S(x', y')]] \implies S(x_1, y')] \),

and \( \mathcal{R}_2 \) is \( R^x(x_1, x_2, x), S^x(x_3, x_2) \) and \( S^2(y_1, y_2) \).

10. \( \forall x, x_1, y, y_1. [\forall y'. [\mathcal{CA}_5 \implies S(x, y')] \text{ and } \mathcal{R}_3 \implies x \leq y] \),

where \( \mathcal{CA}_5 \) is conjunction of

(a) \( \forall x'. [\forall y''. [x' \leq y'' \implies S(x', y')]] \),

(b) \( \forall x''', x'''. [x''' \leq x_1 \text{ and } \forall y'''. [x''' \leq y'''] \implies R(x''', x'''', y'')] \)

and \( \mathcal{R}_3 \) is \( S(x_1, y) \) and \( R^x(x_1, y, y_1) \).

In each case, all (decorated) \( x \) range over \( X \) and all (decorated) \( y \) do over \( Y \).

We have already seen invariance of validity of sequents between polarity frames of distributive modal substructural logics and the prime skeletons in Theorem 5.3. Together with [26, Theorem 5.10], we obtain invariance of first-order correspondents of sequents with consistent variable occurrence.

**Theorem 5.10.** Let \( \phi \equiv \psi \) be a sequent with consistent variable occurrence, and \( \zeta \) the first-order correspondent of \( \phi \equiv \psi \). For each polarity frame of distributive modal substructural logics \( F \) and the prime skeleton \( F_s, F \) is a model of \( \zeta \) if and only if \( F_s \) is a model of \( \zeta \).

\[
\begin{align*}
F \models \phi \equiv \psi & \iff F_s \models \phi \equiv \psi \\
\emptyset & \models \emptyset \\
F \models \zeta & \iff F_s \models \zeta
\end{align*}
\]

**Corollary 5.11.** For every sequent with consistent variable occurrence, the class of prime polarity frames of distributive modal substructural logics which validate the sequent is characterised by first-order sentences.

Unlike invariance of the first-order definability of sequents with consistent variable occurrence between polarity frames of distributive modal substructural logics and prime polarity frames of distributive modal substructural logics, invariance of the first-order definability of sequents which has consistent variable occurrence does not follow directly from invariance of validity of sequents. This is because we have the algorithm to compute first-order correspondents for p-frames only with the frame language for polarity frames, see [26] for the frame language, hence they are not appropriate first-order sentences to describe Kripke frames. To relate these two types of frames, we show the following.

**Proposition 5.12.** For every Kripke frame \( \mathcal{K} \), \( \text{isol}(\text{pol}(\mathcal{K})) \) is isomorphic to \( \mathcal{K} \), i.e. \( \mathcal{K} \cong \text{isol}(\text{pol}(\mathcal{K})) \).
Proof. By definition, there is a natural bijection between $W$ and $W_X$. Furthermore, the preordered sets $\langle W, \leq \rangle$ and $\langle W_X, \leq \rangle$ are isomorphic as we have already seen before. For $R$ and $S$, we can apply the fact that $R^\circ(x_i,x_j,x_k) \iff R_\circ(w_i,w_j,w_k)$ and $S^\circ(x_i,x_j) \iff S_\circ(w_i,w_j)$ in the proof of Theorem 5.6. Finally, the isomorphisms $O \cong O^\circ$ and $N \cong N^\circ$ are by definition. Therefore, $\mathbb{K} \cong \text{isol}(\text{pol}(\mathbb{K}))$.

Remark 5.13. Analogously, we can also prove the similar proposition, that is, for each prime polarity frame $F$, $\text{pol}(\text{isol}(F))$ is essentially isomorphic to $F$, namely the difference is numbers of $\leq$-equivalent elements. However, this is not the proper isomorphism unlike Proposition 5.12.

Proposition 5.14. Let $\mathbb{F} = \langle X,Y,\leq,R,S,O_X,O_Y,N_X,N_Y \rangle$ be a prime polarity frame of distributive modal substructural logics. For every first-order sentence $\zeta$ on the frame language for polarity frames, there is an equivalent first-order sentence $\eta$ on the restricted frame languages on $X$, $\leq$ on $X$, $R^\circ$, $S^\circ$, $O_X$ and $N_X$.

Proof. It suffices to show that there exist equivalent expressions on the limited language for $\leq$ on $X \times Y$, $\leq$ on $Y$, $R$, $R^\circ$, $R^\circ$, $S$, $S^\circ$, $O_Y$ and $N_Y$. For prime polarity frames, we have that $x_i \leq x_j \iff y_i \leq y_j$ and $x_i \leq y_j \iff x_i \not\leq x_j$. Hence, we can obtain equivalent expressions for $\leq$ on $X$ and $\leq$ on $X \times Y$ as $\leq$ by replacing elements in $Y$ by the splitting counterparts in $X$, and $x_i \not\leq x_j$ for $x_i \leq y_j$, respectively. As we have seen in the proof of Theorem 5.6, on prime polarity frames, $R^\circ(x_i,x_j,x_k) \iff R^\circ(x_j,y_i,x_k) \iff R^\circ(y_i,x_j,y_k)$. So, for $R^\circ$ and $R^\circ$, we can take $R^\circ$ with replacing elements in $Y$ by the splitting counterparts in $X$ as an equivalent expression. And, by Proposition 4.11, for $R$, we can take $\neg R^\circ$ (where $\neg$ is negation) replacing elements in $Y$ by the splitting counterparts in $X$ as an equivalent expression. Analogously, for $S$ and $S^\circ$, we can take $\neg S^\circ$ and $S^\circ$ with replacing elements in $Y$ by the splitting counterparts in $X$ as equivalent expressions, respectively. Furthermore, since $x_i \in O_X \iff y_i \not\in O_Y \iff x_i \in O_X \iff y_i \not\in O_Y$ and $x_i \in N_X \iff y_i \not\in N_Y$, we can take $\neg O_X$ and $\neg N_X$ as equivalent expressions, respectively. Therefore, for every first-order sentence $\zeta$ on the frame language for polarity frames, we can find an equivalent first-order sentence $\eta$ on the restricted frame language by substituting the above equivalent expressions and changing the domain $Y$ to the domain $X$.

Finally, we show the main theorem.

Theorem 5.15 (Main). For every formula $\phi$ with consistent variable occurrence, there exists a first-order sentence on the frame language for Kripke frames of distributive modal substructural logics which characterises the class of Kripke frames which validate $\phi$.

Proof. For any Kripke frame $\mathbb{K}$, if $\mathbb{K} \models t \iff \phi$ then by Theorem 5.6 the polarisation $\text{pol}(\mathbb{K})$ is a prime polarity frame. Moreover, by Theorem 5.7, $\text{pol}(\mathbb{K}) \models t \iff \phi$. By the assumption that $\phi$ has consistent variable occurrence, the sequent $t \iff \phi$
also has consistent variable occurrence. Thanks to the Sahlqvist theorem [26, Theorem 5.10], there is a first-order correspondent $\zeta$. By Proposition 5.14, on prime polarity frames, we can find a first-order sentence $\eta$ on the restricted languages on $X$, $\leq$ on $X$, $R^\circ$, $S^\circ$, $O_X$ and $N_X$, which is equivalent to $\zeta$. Note that $\eta$ is a first-order sentence which describe the isolation $isol(poly(K))$. In addition, Proposition 5.4 derives $isol(poly(K)) \models \phi$. Finally, since $K \cong isol(poly(K))$ by Proposition 5.12, we conclude that $\eta$ is a first-order correspondent of $\phi$ on Kripke frames.

For sequents which have consistent variable occurrence in Examples 5.8 and 5.9, we can obtain the first-order correspondents on Kripke frames by applying Proposition 5.14.

**Example 5.16.** The first-order correspondents on Kripke frames for sequents in Example 5.8 are:

1. $O(w)$ and $\neg N(w)$, i.e. $O = W$ and $N = \emptyset$.
2. $R_\circ(w, w, w)$.
3. $R_\circ(w_1, w_2, w) \implies R_\circ(w_2, w_1, w)$.
4. $v_1 \not\preceq w$ and $R_\circ(w, v, v_1) \implies v_1 \preceq v$.
5. $R_1' \implies v \not\preceq w$, where $R_1'$ is $\neg R_\circ(w_2, w, v_2), \neg R_\circ(w_2, w_1, v_2), R_\circ(w_2, v_1, v_2)$ and $R_\circ(w_1, v, v_1)$.
6. $CA_1' \implies v \not\preceq w$, where $CA_1'$ is $\forall w_1. [\forall w_2, v_1. [v \not\preceq w_2 \implies \neg R_\circ(w_2, w_1, v_1)] \implies \neg R_\circ(w_1, v, v)]$.
7. $CA_2', R_\circ(w_2, w_3, w_1)$ and $S_\circ(w_1, w) \implies \neg R_\circ(w_4, w, v_1)$, where $CA_2'$ is conjunction of
   (a) $\neg S_\circ(w_2, v_1)$,
   (b) $\forall w', w''. [O(w') \land R_\circ(w', w_1, w'') \implies \neg S_\circ(w'', v_1)]$,
   (c) $\forall w'. [S_\circ(w_4, w') \implies N(w')]$.
8. $CA_3'$ and $R_\circ(w_1, v_1, v_2) \implies \neg S_\circ(w, v_1)$, where $CA_3'$ is $\forall w', v'. [CA_3'(a) \land CA_3'(b) \implies \neg R_\circ(w', w, v')]$ with
   (a) $CA_3'(a): \forall v''. [v'' \not\preceq w_1 \implies \neg S_\circ(w', v'')]$,
   (b) $CA_3'(b): \forall w''. [v' \not\preceq w'' \implies \neg S_\circ(w'', v')]$.
9. $CA_4'$ and $R_2' \implies \neg R_\circ(w, w_4, v_1)$, where $CA_4'$ is $\forall v'. [\forall w'. [\neg S_\circ(w', v_2) \implies \neg S_\circ(w', v')] \implies \neg S_\circ(w_4, v_1)]$, and $R_2'$ is of $R_\circ(w_1, w_2, w), S_\circ(w_3, w_2)$ and $S_\circ(v_1, v_2)$.
10. $\forall v'. [CA_5' \implies \neg S_\circ(w, v')]$ and $R_3' \implies v \not\preceq w$, where $CA_5'$ is conjunction of

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sentences are equivalent to the following:

(a) $\forall w'. [\forall v'''. [v''' \not\leq w'] \implies \neg S_5(w', v')]$,
(b) $\forall w'', w'''. [w'' \leq w_1 \text{ and } \forall v''''. [v''''. \not\leq w'''] \implies \neg R_6(w'', w''', v''')]$,

and $R'_3$ is $\neg S_5(w_1, v)$ and $R_6(w_1, v, v_1)$.

In each case, all (decorated) $w$ are universally quantified over the whole sentence, and all (decorated) $w$ and $v$ range over $W$. One may feel that some first-order correspondents in the above list look far from the Sahlqvist correspondents in modal logic, at first. However, we also mention that the above first-order sentences are equivalent to the following:

4'. $R_6(w, v, v_1) \implies v_1 \leq w$ or $v_1 \leq v$: this is the so-called Dunn's postulate (note that the order is now dual of the original frame).

5'. $R_6(w_2, v_1, v_2), R_6(w_1, v, v_1)$ and $v \leq w \implies R_6(w_2, w, v_2)$ or $R_6(w_2, w, v_1)$. Note that this solves an open question listed in [15].

6'. $v \leq w \implies \exists w_2, v_1. [R_6(w_2, w, v_1) \implies v \leq w]$ and $R_6(w_1, w, v_2)$.

7'. $R''_1$ and $\forall w'. [S_5(w_4, w') \implies N(w')] \implies CC_1$, where $CC_1$ is disjunction of

(a) $S_5(w_2, v_1)$,
(b) $\exists w', w''. [O(w'), R_6(w', w_1, w'')$ and $S_5(w'', v_1)]$,

and $R''_1$ is $S_5(w_1, w), R_6(w_2, w_3, w_1)$ and $R_6(w_4, w, v_1)$.

8'. $S_5(w, v_1)$ and $R_6(w_1, v_1, v_2) \implies \exists w', v', [CC_2]$,

where $CC_2$ is conjunction of

(a) $\forall w'. [S_5(w', v') \implies v' \leq w_1]$,
(b) $\forall v'''. [S_5(w'', v') \implies v' \leq w'']$,
(c) $R_6(w', w, v')$.

9'. $R_6(w_1, v_2, w_1), S_5(w_3, v_2), S_5(v_1, v_2)$ and $R_6(w, w_1, v_1) \implies CC_3$, where $CC_3$ is $\exists v'. [\forall w'. [S_5(w', v') \implies S_5(w', v_2)]$ and $S_5(w_1, v')]$.

10'. $v \leq w, R_6(w_1, v, v_1)$ and $\forall v'. [S_5(w, v') \implies CC_4] \implies S_5(w_1, v)$,

where $CC_4$ is disjunction of

(a) $\exists w'''. [\forall v'''. w'''. \not\leq w']$ and $S_5(w', v')]$,
(b) $\exists w'', w'''. [w'' \leq w_1, \forall v'''. w'''. \not\leq w''']$ and $R_6(w'', w''', v')$.

In each case, all (decorated) $w$ are universally quantified over the whole sentence, and all (decorated) $w$ and $v$ range over $W$.

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6 Application: intuitionistic modal logic

In [8], the Sahlqvist canonicity theorem for intuitionistic modal logic has been shown. On the other hand, as [8] had mentioned in a footnote, the first-order definability on Kripke frames is unclear so far. However, as we have seen in the previous section, we can prove the first-order definability on Kripke frames via the Sahlqvist theorem on bi-approximation semantics [26]. In this section, we apply the first-order definability in the previous section to intuitionistic modal logic. The language consists of almost the same language as distributive modal substructural logics. Hence, we take each Kripke frame for intuitionistic modal logic as follows: for intuitionistic modal logic is almost the same as the satisfaction relation for distributive modal substructural logics. The only exceptions are the polarisation, and interpret it to one on the restricted frame language.

Let us consider the first-order definability on Kripke frames for intuitionistic modal logic as follows. To apply our main theorem (Theorem 5.15) for intuitionistic modal logic, we firstly look at each Kripke frame for intuitionistic modal logic as a Kripke frame of distributive modal substructural logics as follows: for each Kripke frame for intuitionistic modal logic \( \mathcal{K} = \langle W, \preceq, S_\Diamond \rangle \), we let, for all \( w_i, w_j, w_k \in W \), \( S_\Diamond(w_i, w_j, w_k) \) be \( w_k \preceq w_i \) and \( w_k \preceq w_j \), \( O := W \) and \( N := \emptyset \). Then, the sextuple \( \langle W, \preceq, S_\Diamond, O, N \rangle \) is a Kripke frame of distributive modal substructural logics, which validate all sequents in the intuitionistic modal logic, hence the polarisation also validates all sequents in the intuitionistic modal logic, see also [26].

Next we take the same strategy as Theorem 5.15, namely given a formula \( \phi \) with consistent variable occurrence, we calculate the first-order correspondent on the polarisation, and interpret it to one on the restricted frame language. Finally, we reinterpret the first-order correspondent on the restricted frame language by replacing \( S_\Diamond(w_i, w_j, w_k) \) by \( w_k \preceq w_i \) and \( w_k \preceq w_j \), \( O \) by \( W \) and \( N \) by \( \emptyset \). Notice that we can also replace \( w_i \) by \( w_k \) in \( S_\Diamond(w_i, w_j, w_k) \) to simplify sentences, because \( S_\Diamond \) only appears for \( \rightarrow \) in our current setting: see also Example 6.2.

Ghilardi and Meloni in [8] discussed the first-order definability of sequents which have consistent variable occurrence. They say that the first-order definability is clear only for the limited cases. In fact, the sequents of item 8, item 9 and item 10 in Example 5.8 are of the unclear cases. In the rest of this section, as examples of our theory, we show the first-order correspondents of the
sequents (items 7 - 10) in Example 5.8. To do so, we simplify them into the following equivalent ones in intuitionistic modal logic.

**Example 6.2.** The following sequents are obtained by replacing $\phi \circ \psi$ by $\phi \land \psi$, $\psi \leftarrow \phi$ by $\phi \rightarrow \psi$, $t$ by $\bot \rightarrow \bot$, and $f$ by $\bot$ syntactically, and by using logical equivalence on intuitionistic modal logic.

7. $\Diamond(p \land q) \Rightarrow \Box \bot \rightarrow \Diamond p$: there is no critical subformula.

8. $\Box p \Rightarrow \Diamond q \Rightarrow \Box(p \rightarrow q)$: the only critical subformula is $\Box p \rightarrow \Diamond q$.

9. $\Box \Diamond p \land \Box \Diamond q \Rightarrow \Box \Diamond q \rightarrow \Box \Diamond p$: the critical subformulae are $\Box \Diamond p$ in the left formula and $\Box \Diamond q$.

10. $\Box(\Diamond p \lor (q \land p)) \Rightarrow (q \rightarrow p) \lor \Diamond q$: the critical subformula is $\Box(\Diamond p \lor (q \land p))$ only.

Now, by applying our theory for the above sequents, we obtain the following first-order correspondents.

**Example 6.3.**

7. $S_\Diamond(w, v)$ and $\forall w_1. [\neg S_\Diamond(v, w_1)] \rightarrow S_\Diamond(w, v)$. Notice that this first-order correspondent is tautology. However it does not contradict. Actually, the sequent $\Diamond(p \land q) \Rightarrow \Box \bot \rightarrow \Diamond p$ is in the intuitionistic modal logic (provable in the system).

8. $S_\Diamond(w, v) \rightarrow \exists v_1. [CC'_1]$, where $CC'_1$ is conjunction of

(a) $v_1 \preceq w$,
(b) $\forall v_2. [S_\Diamond(v_1, v_2) \rightarrow v_2 \preceq v]$,
(c) $\forall w_1. [S_\Diamond(w_1, v_1) \rightarrow v \preceq w_1]$.

9. $S_\Diamond(w_1, w), S_\Diamond(v, v_1), v \preceq w$ and $CC'_2 \rightarrow \exists v_3. [CC'_3]$, where $CC'_2$ is $\forall w_2. [S_\Diamond(v, v_2) \rightarrow \exists w_2. [S_\Diamond(w_2, v_2)]]$, and $CC'_3$ is $\forall w_3. [S_\Diamond(w_3, v_3) \rightarrow S_\Diamond(w_3, v_1)]$ and $S_\Diamond(w, v_3)$.

10. $v \preceq w$ and $\neg S_\Diamond(v, w) \rightarrow \exists v_1. [CC'_4]$, where $CC'_4$ is conjunction of

(a) $S_\Diamond(w, v_1)$,
(b) $\forall w_1. [S_\Diamond(w_1, v_1) \rightarrow v \preceq w_1]$,
(c) $(S_\Diamond(v_1, v) \lor v \preceq v_1)$.

In each case, all (decorated) $w$ are universally quantified over the whole sentence, and all (decorated) $w$ and $v$ range over $W$. Note that each sentence is an equivalently simplified one, “by hand.”
7 Conclusion

In this paper, we have investigated the first-order definability of sequents (formulae) which have consistent variable occurrence on Kripke semantics over distributive modal substructural logics. By introducing the prime skeletons and extending the results of distributive polarity frames in [21], we have noticed that the prime skeletons provide a consistent bridge to nicely connect Kripke semantics to bi-approximation semantics. In addition, we have shown invariance of validity of sequents by introducing the back-and-forth construction between Kripke semantics and bi-approximation semantics. Throughout the back-and-forth construction, we have supplied a gateway to transfer the first-order definability on bi-approximation semantics to Kripke semantics. As a result, we can provide the first-order definability of all sequents (formulae) with consistent variable occurrence, which had not been thoroughly solved in [8].

As a final remark, note that the gateway we constructed in this paper is quite possible to be useful to transfer other logical properties as well. That is, we may be able to find some logical properties which are provable in modal logic but not yet clear how to generalise to distributive modal substructural logics via bi-approximation semantics.

References


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